Universality behind Basquin's Law of Fatigue

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(Received 15 May 2007; published 4 March 2008)

Basquin's law of fatigue states that the lifetime of the system has a power-law dependence on the external load amplitude, $t_f \sim \sigma_0^{-\alpha}$, where the exponent α has a strong material dependence. We show that in spite of the broad scatter of the exponent α, the fatigue fracture of heterogeneous materials exhibits universal features. We propose a generic scaling form for the macroscopic deformation and show that at the fatigue limit the system undergoes a continuous phase transition. On the microlevel, the fatigue fracture proceeds in bursts characterized by universal power-law distributions. We demonstrate that the system dependent details are contained in Basquin's exponent for time to failure, and once this is taken into account, remaining features of failure are universal.

DOI: 10.1103/PhysRevLett.100.094301 PACS numbers: 46.50.+a, 62.20.M−, 61.82.Pv

Disordered media subject to subcritical external loads present a time dependent macroscopic response and typically fail after a finite time [1]. Such time dependent fracture evidently plays a crucial role in a large variety of physical, biological, and geological systems, such as the rupture of adhesion clusters of cells in biomaterials under external stimuli [2], the subcritical crack growth due to thermal activation of crack nucleation [3,4], creep [5] and fatigue fracture of materials [6,7], and the emergence of earthquake sequences [8]. One of the most important scaling laws of time dependent fracture is the empirical Basquin law of fatigue which states that the lifetime t_f of samples increases as a power law when the external load amplitude σ_0 decreases, $t_f \sim \sigma_0^{-\alpha}$ [9]. The measured values of the Basquin exponent α typically vary over a broad range indicating a strong dependence on material properties [9–11].

In this Letter we study the fatigue fracture of heterogeneous materials focusing on the underlying microscopic mechanism of the fatigue process and its relation to the macroscopic time evolution. We develop two generic models of time dependent fracture, namely, a fiber bundle model and a discrete element approach, which both capture the most important ingredients of the fatigue failure of disordered materials. Analytic solutions and computer simulations reveal that the models recover the Basquin law of fatigue, whose exponent is determined by the damage process. We show that, as a consequence of healing, a finite fatigue limit emerges at which the system undergoes a continuous phase transition from a regime where macroscopic failure occurs at a finite time to another one exhibiting only partial failure in the system having an infinite lifetime. Based on analytic solutions, we propose a generic scaling form for the macroscopic deformation. On the microlevel the fatigue of the material is accompanied by an avalanche activity where bursts of local breakings are triggered by damage sequences. We demonstrate analytically that the microscopic bursting activity underlying fatigue fracture is characterized by universal power-law distributions which implies that the nonuniversality of the Basquin exponent at the macro-level is solely due to the specific degradation process of the material.

First we consider a mean-field model of fatigue fracture, namely, a fiber bundle model (FBM) where fibers fail either due to immediate breaking or to ageing [12]. For the load redistribution after failure events, equal load sharing is assumed so that all the fibers carry the same load [13]. During the evolution of the system, a fiber breaks instantaneously at time t when the load on it $p(t)$ exceeds the local tensile strength p_{th}^i $(i = 1, \ldots, N)$. All intact fibers accumulate damage $c(t)$ due to the load $p(t)$ that they have experienced and break when $c(t)$ exceeds the local damage threshold c_{th}^i $(i = 1, \ldots, N)$. The accumulated damage $c(t)$ up to time t is obtained by integrating over the entire loading history of the specimen $c(t) = a\int_0^t e^{-((t-t')/\tau)} p(t')^\gamma dt'$, where $a > 0$ is a scale parameter, while the exponent $\gamma > 0$ controls the rate of damage accumulation [10,11]. To capture damage recovery in the model due to healing of microcracks [10] or thermally activated rebinding of failed contacts [2,8], we introduce a memory term in the above damage law of exponential form whose characteristic time scale τ defines the memory range of the system [2,4,8]. Hence, during the time evolution of the bundle, the damage accumulated over the time interval $t' < (t - \tau)$ heals. Assuming independence of the two breaking thresholds p_{th} and c_{th}, the macroscopic evolution of the system under a constant external load σ_0 can be cast into the form

$$\sigma_0 = \{1 - F[c(t)]\}\{1 - G[p(t)]\}p(t), \qquad (1)$$

where G and F denote the cumulative distributions of p_{th} and c_{th}, respectively. We solved Eq. (1) analytically obtaining the load $p(t)$ on the intact fibers at a constant external load $\sigma_0 < \sigma_c$, with the initial condition $p(t = 0) = p_0$, where p_0 is the solution of the constitutive equation $\sigma_0 = [1 - G(p_0)]p_0$ [13]. Here σ_c denotes the strength of the material. The most important input parameters of the model calculations are a, γ and τ.

On the macrolevel the process of fatigue is characterized by the evolution of deformation $\varepsilon(t)$ of the specimen, which is related to $p(t)$ as $p(t) = E\varepsilon(t)$, where $E = 1$ is the Young modulus of fibers. Neglecting immediate breaking and healing, for uniformly distributed threshold values the exact solution of the equation of motion Eq. (1) reads

$$\varepsilon(t) = \sigma_0[(t_f - t)/t_f]^{-1/(1+\gamma)} \quad \text{and} \quad t_f = \frac{\sigma_0^{-\gamma}}{a(1+\gamma)}, \tag{2}$$

where t_f denotes the lifetime of the system. Equation (2) shows that damage accumulation leads to a finite time singularity where the deformation $\varepsilon(t)$ of the system has a power-law divergence with an exponent determined by γ. It is important to emphasize that t_f has a power-law dependence on the external load σ_0 in agreement with Basquin's law of fatigue found experimentally in a broad class of materials [9–11]. The Basquin exponent of the model therefore coincides with that of the microscopic degradation law $\alpha = \gamma$. Another interesting outcome of the derivation is that the macroscopic deformation $\varepsilon(t)$ of a specimen undergoing fatigue fracture obeys the generic scaling form $\varepsilon(t) = \sigma_0^{\delta} S(t\sigma_0^{\beta})$, where the scaling function S has the property $S(t\sigma_0^{\beta}) \sim (t_a - t\sigma_0^{\beta})^{-1/(1+\gamma)}$, with $t_a = a(1+\gamma)$ and the scaling exponents are $\delta = 1$ and $\beta = \gamma$. Figure 1 presents a verification of this scaling law on experimental data from asphalt specimens obtained at two different load values [12]. The good quality data collapse obtained by rescaling the two axis and the power-law behavior of S as a function of the time-to-failure demonstrates the validity of our scaling relation.

Healing dominates if for a fixed load σ_0 the memory time τ is smaller than the lifetime obtained without healing $\tau \lesssim t_f(\sigma_0, \tau = +\infty)$. Then, a threshold load σ_l emerges below which the system relaxes; i.e., the deformation $\varepsilon(t)$ converges to a limit value with a characteristic relaxation time t_r resulting in an infinite lifetime. Figure 2 presents the characteristic time scale of the system varying the external load over a broad range. The results from numerical simulations with the complete FBM (i.e., including immediate breaking and healing) are in excellent agreement with the measured lifetime of asphalt samples for $\sigma_0 > \sigma_l$, recovering also the Basquin exponent [12]. The regime below σ_l is of particular importance in geodynamics where memory effects take place during cyclic loading of rocks with a stress amplitude increasing from one cycle to the next [14]. It is important to note that approaching the fatigue limit σ_l from either side, the characteristic time scale diverges. Figure 3 shows that both the relaxation time t_r and the lifetime t_f follow a power law as a function of the difference from the fatigue limit with distinct exponents: $t_r \sim (\sigma_l - \sigma_0)^{-1/3}$ and $t_f \sim (\sigma_0 - \sigma_l)^{-2/3}$. We stress that the exponents neither depend on the disorder distributions (F and G) nor on the details of the damage law (a, γ, and τ); i.e., they are universal, implying a continuous phase transition at the fatigue limit σ_l between partial failure and macroscopic fracture (see Fig. 3).

Our calculations revealed that the Basquin law of lifetime emerges on the macrolevel as a consequence of the competition between the two microscopic failure mecha-

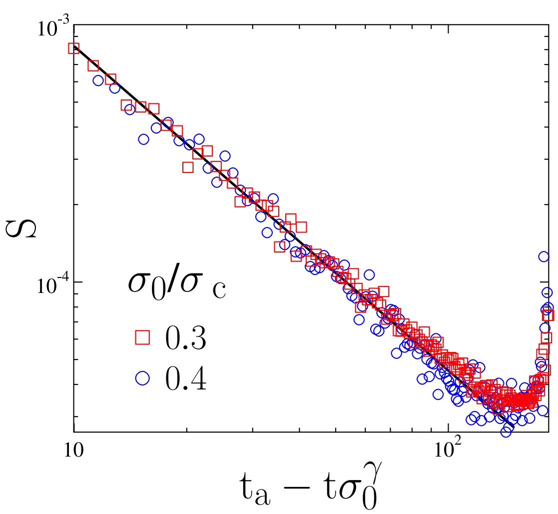

FIG. 1 (color online). Scaling plot of deformation-time curves measured experimentally on asphalt specimens [12]. The scaling function S, obtained by rescaling the two axis, has a power-law dependence on the time-to-failure.

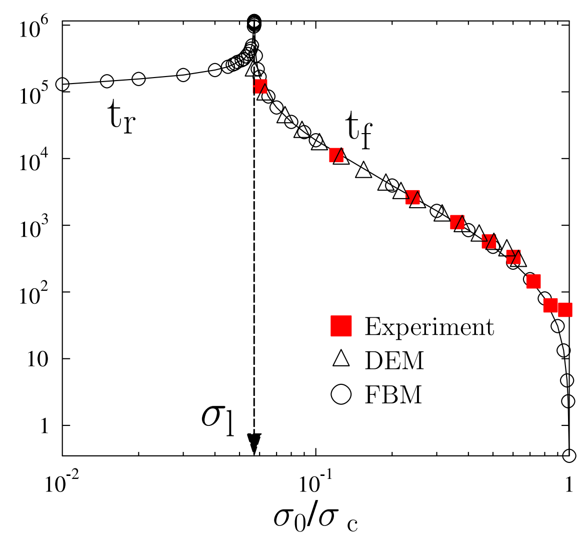

FIG. 2 (color online). Characteristic time scales t_r and t_f of the system. The complete FBM corresponding to Eq. (1) which includes immediate breaking and healing is solved numerically. For $\sigma_0 > \sigma_l$ we see Basquin's law and both models provide a very good fit of the lifetime data of asphalt. The fatigue limit σ_l is indicated by the vertical dashed line.

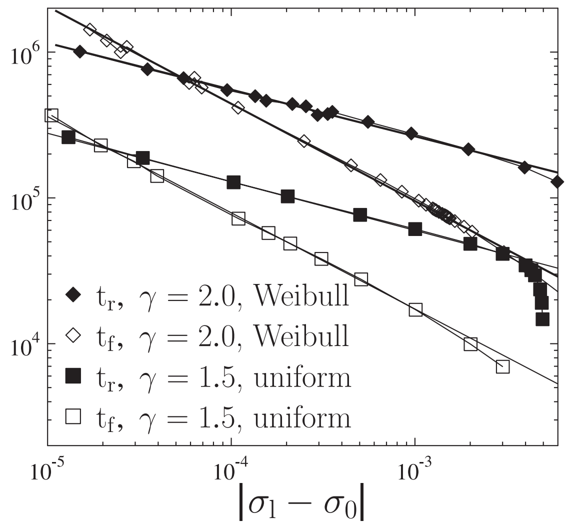

FIG. 3. The relaxation time t_r and lifetime t_f as a function of $|\sigma_l - \sigma_0|$ for different disorder distributions and γ exponents. The straight lines have slopes $-1/3$ and $-2/3$.

nisms of fibers. Rewriting Eq. (1) in the form of the constitutive equation of simple FBMs as $\sigma_0/\{1 - F[c(t)]\} = \{1 - G[p(t)]\}p(t)$ it can be seen that the slow damage process on the left-hand side quasistatically increases the load on the system: ageing fibers accumulate damage and break slowly one by one in the increasing order of their damage thresholds $c^i_{\rm th}$. After a number Δ_d of damage breakings, the emerging load increment on the remaining intact fibers can trigger a burst of immediate breakings. Since load redistribution and immediate breaking occur on a much shorter time scale than damage accumulation, the entire fatigue process can be viewed on the microlevel as a sequence of bursts of immediate breakings triggered by a series of damage events happening during waiting times T, i.e., the time intervals between the bursts. The microscopic failure process is characterized by the size distribution of bursts $P(\Delta)$, damage sequences $P(\Delta_d)$, and by the distribution of waiting times $P(T)$. At small loads $\sigma_0 \ll \sigma_c$ most of the fibers break in long damage sequences, because the resulting load increments do not suffice to trigger bursts. Increasing σ_0 the total number of bursts n_b increases linearly $n_b \sim \sigma_0$ and a power-law regime of burst sizes emerges $P(\Delta) \sim \Delta^{-\xi}$ with the well-known mean-field exponent of FBM $\xi = 5/2$ [15]. When macroscopic failure is approached $\sigma_0 \to \sigma_c$ the failure process accelerates such that the size Δ_d and duration T of damage sequences decrease, while they trigger bursts of larger sizes Δ, and, finally, macroscopic failure occurs as a catastrophic burst of immediate failures. Since in the limiting case of $\sigma_0 \to \sigma_c$ a large number of weak fibers breaks in the initial burst, we found that the distribution $P(\Delta)$ has a crossover to a smaller exponent $\xi = 3/2$, in agreement with Ref. [15]. All these results are independent of γ, a, and τ.

Since damage events increase the load on the remaining intact fibers until an immediate breaking is triggered, the size of damage sequences Δ_d is independent of the damage characteristics $c(t)$ and $F(c_{\rm th})$ of the material, instead, it is determined by the load bearing strength distribution $G(p_{\rm th})$ of fibers. Under broad conditions this mechanism leads to an universal power-law form with an exponential cutoff $P(\Delta_d) \sim \Delta_d^{-1}\exp(-\Delta_d/\langle\Delta_d\rangle)$, where $\langle\Delta_d\rangle \sim \sigma_0^{-1}$. The damage law $c(t)$ of the material controls the time scale of the process of fatigue fracture through the temporal sequence of single damage events. In damage sequences fibers break in the increasing order of their damage thresholds $c^i_{\rm th}$ which determine the time intervals Δt between consecutive fiber breakings. Analytic calculations showed that $P(\Delta t)$ has an explicit dependence on γ as $P(\Delta t) \sim \Delta t^{-(1-1/\gamma)}$; however, the duration of sequences $T = \sum_{j=1}^{\Delta_d} \Delta t_j$, i.e., the waiting times between bursts, follow a universaly power-law distribution $P(T) \sim T^{-1}\exp(-T/\langle T\rangle)$, where only the cutoff has γ dependence $\langle T\rangle \sim \sigma_0^{-(1+\gamma)}$ (see Fig. 4).

The macroscopic lifetime t_f of a finite system can be related to characteristic quantities of the microscopic failure process as $t_f = \sum_{i=1}^{n_b} T_i$, from which the average lifetime can be obtained in the form $t_f \approx \langle n_b\rangle\langle T\rangle$. In the load regime where the generic scaling laws of the distributions $P(\Delta)$, $P(\Delta_d)$, and $P(T)$ prevail, this leads to the form $t_f \sim \sigma_0^{-\gamma}$ in agreement with the Basquin law Eq. (2) of the system. The results demonstrate that the Basquin law of lifetime on the macroscale is a fingerprint of the scale-free microscopic bursting activity, with the material dependence entering only through the damage law determining the waiting times between bursts.

To study the effect of stress concentration and crack growth in fatigue fracture, we also developed a discrete

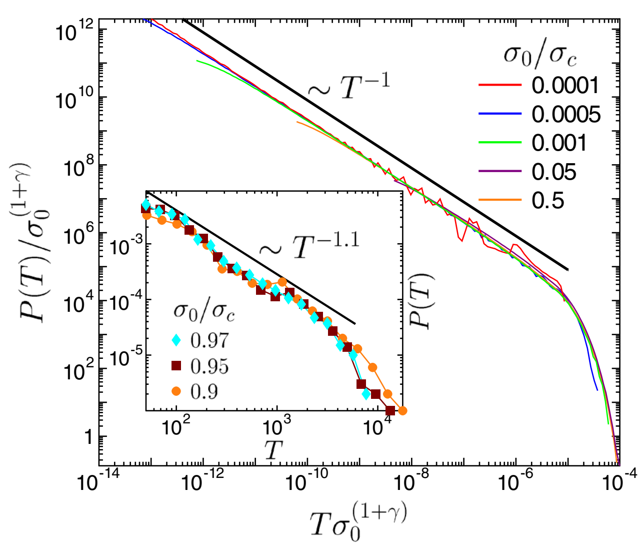

FIG. 4 (color online). Rescaled plot of waiting time distributions $P(T)$ obtained by FBM simulations. Inset: Corresponding results of two-dimensional DEM simulations.

element model (DEM) [16] in which we discretize a two-dimensional disc-shaped specimen in terms of randomly shaped convex polygons connected by elastic beams. The beams fail either due to immediate breaking or damage which are coupled in a single failure variable $q(t) = p(t) + a \int_0^t e^{-(t-t')/\tau} p(t')^\gamma dt'$. Here $p(t)$ describes the deformation state of the beam taking into account both stretching and bending $p(t) = (\varepsilon/\varepsilon_{\rm th})^2 + \max(|\Theta_1|, |\Theta_2|)/\Theta_{\rm th}$, being ε the longitudinal deformation, Θ_1 and Θ_2 the bending angles at the two ends of the beam, and $\varepsilon_{\rm th}$ and $\Theta_{\rm th}$ denote the threshold values a beam can sustain under stretching and bending, respectively. As a consequence, the parameters a, γ, and τ play the same role as their counterparts in our FBM. The time evolution of the system is followed by numerically solving the equations of motion of polygons. The breaking criterion $q(t) > 1$ is evaluated at each time step and beams which fulfil the condition are removed [16]. We study the fatigue fracture under diametric compression of discs with constant stress σ_0 (Brazil test). Figure 2 shows that DEM provides also an excellent fit of the lifetime data of asphalt specimens [12]. DEM simulations revealed that in the presence of stress concentrations bursts are spatially correlated and they can be identified as sudden advancements of slowly growing cracks. DEM results on burst characteristics also show power-law behavior as the mean-field FBM, but with different exponents due to the two-dimensionality of the model. The localized stress concentration built up around cracks gives rise to higher values of the exponents of the size distribution of bursts $P(\Delta) \sim \Delta^{-2.7}$, and of damage sequences $P(\Delta_d) \sim \Delta_d^{-1.8}$, while for the waiting time distribution $P(T)$ the DEM exponent falls very close to the mean-field value (see Fig. 4). The results proved to be independent of the value of γ.

Although the exponent of Basquin's law depends on the microscopic damage accumulation, we found an astonishing spectrum of universal features hidden behind this originally empirical law. We discovered in the experimentally relevant situation of finite damage memory a continuous phase transition between partial failure and macroscopic rupture. On the microscopic level of individual breaking events we showed that the separation of time scales of the two failure mechanisms leads to a bursting activity, where we disclosed several universal scaling laws in the distributions and determined their exponents as well in mean-field as in two dimensions. In summary our approach provides a direct connection between the microscopic mechanisms constituting the main ingredients of the model (i.e., immediate breaking, damage accumulation and healing of microcracks) and the macroscopic behavior of the fatigue process. The (macroscopic) Basquin exponent coincides with the (microscopic) exponent of the degradation law, namely $\alpha = \gamma$. Our methodology is also capable to show explicitly the bridge between the (universal) mechanism related with the scale-free bursting activity at the micro-scale and the (nonuniversal) lifetime law of the material at the macroscale.

This work opens up new experimental challenges. Our scaling relation of the macroscopic deformation should be verified on various types of materials, after which it could help to extract the relevant information from fatigue life measurements. In the infinite lifetime limit, $\sigma_0 \lesssim \sigma_l$, the experimental confirmation of the power-law variability with load of the relaxation time should certainly provide some considerable insight on the role of healing in the entire fatigue process. For similar reasons, it would be also interesting to verify the distinct lifetime behavior obtained from the model in the other limit of low external loads, $\sigma_0 \gtrsim \sigma_l$. On the microscopic level both the size Δ_d of damage sequences and magnitude T of waiting times between bursts should obey universal power-law distributions that might reflect the intrinsic features of the typical restructuring events taking place at the microscopic level. Acoustic emission measurements could be conducted in conjunction with fatigue experiments to confirm our claim for universality behind Basquin's law.

We thank the Brazilian agencies CNPq, CAPES, FUNCAP, and FINEP, and the Max Planck prize for financial support. F. Kun was supported by OTKA No. T049209.

[1] M. Alava, P. K. Nukala, and S. Zapperi, Adv. Phys. **55**, 349 (2006).
[2] T. Erdmann and U. Schwarz, Phys. Rev. Lett. **92**, 108102 (2004).
[3] S. Santucci *et al.*, Phys. Rev. Lett. **93**, 095505 (2004).
[4] D. Sornette and G. Ouillon, Phys. Rev. Lett. **94**, 038501 (2005).
[5] H. Nechad *et al.*, Phys. Rev. Lett. **94**, 045501 (2005); R. C. Hidalgo, F. Kun, and H. J. Herrmann, Phys. Rev. E **65**, 032502 (2002).
[6] D. Sornette and C. Vanneste, Phys. Rev. Lett. **68**, 612 (1992); H. J. Herrmann, J. Kertész, and L. de Arcangelis, Europhys. Lett. **10**, 147 (1989).
[7] D. Farkas, M. Willemann, and B. Hyde, Phys. Rev. Lett. **94**, 165502 (2005); D. Sornette, T. Magnin, and Y. Brechet, Europhys. Lett. **20**, 433 (1992).
[8] C. Marone, Nature (London) **391**, 69 (1998).
[9] O. H. Basquin, *Proceedings of American Society of Testing Materials ASTEA* (10), 625 (1910).
[10] D. Krajcinovic, *Damage Mechanics* (Elsevier, Amsterdam, 1996); S. Suresh, *Fatigue of Materials* (Cambridge University Press, Cambridge, England, 2006).
[11] A. Rinaldi *et al.*, Int. J. Fatigue **28**, 1069 (2006); M. E. Biancolini *et al.*, Int. J. Fatigue **28**, 1820 (2006).
[12] F. Kun *et al.*, J. Stat. Mech. (2007) P02003; M. J. Alava, J. Stat. Mech. (2007) N04001.
[13] J. V. Andersen, D. Sornette, and K. Leung, Phys. Rev. Lett. **78**, 2140 (1997); R. C. Hidalgo *et al.*, Phys. Rev. Lett. **89**, 205501 (2002).
[14] A. Lavrov, Int. J. Rock Mech. Min. Sci. **40**, 151 (2003).
[15] S. Pradhan, A. Hansen, and P. C. Hemmer, Phys. Rev. Lett. **95**, 125501 (2005).
[16] H. A. Carmona *et al.*, Phys. Rev. E **75**, 046115 (2007).