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Damage growth in fibre bundle models with localized load sharing and environmentally-assisted ageing

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Abstract. One of the most fundamental models for the complex behaviour of damage accumulation in earth materials is the fibre bundle model. One distinguishes between models with localized load sharing (LLS) and models with equal load sharing (ELS). While ELS models can be treated by mean field theory, the behaviour of LLS models is usually more complicated. Here, we consider a fibre bundle model with LLS where, in addition, we introduce a time scale by incorporating time dependent ageing of the fibres due to the accumulation of damage driven by the locally acting stress in a chemically active environment. If the accumulated damage exceeds a random threshold, the fibres fail. The non-trivial time dependence of the cumulative damage in the system can be attributed to different mechanisms that dominate at different time scales. We include this information into an analytical description of the damage accumulation process and show that the analytical description is in agreement with numerical results.

1. Introduction

The study of subcritical fracture processes is an important issue in physics, engineering and geosciences. One of the most fundamental models for the failure of heterogeneous materials is the fibre bundle model (FBM) [1, 2] which envisages a material as a set of parallel fibres with identical elastic properties but with stochastic failure thresholds [1–5]. If a fibre breaks as the local stress exceeds its failure threshold σ_{th}, the stress carried by this fibre is redistributed. In equal load sharing (ELS) models, stress is equally redistributed over all remaining intact fibres, which allows to treat these models by mean field theory. In localized load sharing (LLS) models, on the other hand, redistribution is locally to the close neighborhood of the broken fiber and the behavior is more complicated to analyze. In physical terms, LLS fiber bundle models describe materials failure in situations where the range of internal stresses is limited, e.g. by the vicinity of a free surface [6, 7]. To model subcritical failure, the FBM has been extended by incorporating an additional ageing of the fibres [8–12] which accounts for chemical weakening effects associated with the environment. To this end one introduces a damage variable c, and fibres now break either when their stress reaches the threshold σ_{th} or when their damage reaches a threshold c_{th}.

In the present study we consider a FBM on a square lattice with LLS, redistributing the load equally over all intact nearest neighbors of a broken fiber. Regarding the stress-driven ageing process, we assume for the accumulated damage in each fibre a time-dependence of the form

$$c = a \int_0^t \sigma(t')dt' \tag{1}$$

where $\sigma(t')$ is the time dependent local stress. For σ_{th} we assume a uniform distribution between 0 and 1, and for c_{th} a uniform distribution between $1-W$ and $1+W$. By varying W in the range $0 \leq W \leq 1$ we control the amount of damage disorder.

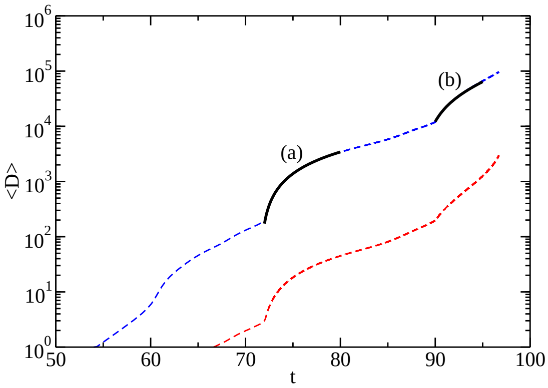

Figure 1. Average damage $\langle D \rangle$ initiated by ageing (dashed blue line) or by local stress redistribution (dashed red line) for a 401×401 sized lattice with $a = 1$, $W = 0.1$ and $\sigma = 0.01$ as a function of time t. The full black curves show the corresponding analytical results from (a) Eq. (6) and (b) Eq. (11) with $d = 146$ and $d' = 2150$ for the respective time periods.

Figure 1 shows the time dependent average damage on a $N = 401 \times 401$ lattice with applied constant stress $\sigma = 0.01$ and constants $a = 1$ and $W = 0.1$. Failures triggered by ageing (dashed blue line) and by local stress redistribution (dashed red line) are plotted separately. One clearly sees that the damage makes significant transient jumps and the functional form of the damage curve seems to be nontrivial. One also sees that in the considered time window failures triggered by local stress redistribution only play a minor role in comparison to those triggered by the ageing process. In the following we focus on this dominant part of the damage and derive analytical equations describing the onset and the functional form of two of the observed jumps. These jumps originate from a sequence of different mechanims that dominate the damage process: In the very beginning of a simulation, a certain fraction of fibres break as we apply the constant stress. Their neighbours thus experience higher local stresses and fail first. Later, also fibres with intact neighbourhood start to fail due to ageing and this process then dominates the system.

2. Breaking of fibres with one broken neighbour

If the number of fibres is large, the smallest threshold for failure by aging has the approximate value $c_{th} \approx 1 - W$. The local stress acting on a fibre with one broken neighbour is $5\sigma/4$. Since the local stress will not increase until other fibres break, the onset time t_1 for failure of fibres with one broken neighbour is given by $1 - W = 5a\sigma t_1/4$, which is equivalent to

$$t_1 = 4(1 - W)/(5a\sigma). \tag{2}$$

Since the critical threshold values for the local stresses are equally distributed between 0 and 1, the number of initially broken fibres can be approximated by σN. Each of these broken fibres has 4 neigbours to which the local stress is redistributed. Thus the total number of fibres with one broken neighbor can initially be approximated by $4N\sigma$. On average all these fibres will have equal distributed thresholds between $1-W$ and $1+W$. Thus the average distance between two consecutive thresholds is $W/(2N\sigma)$. After the breaking of i fibres, on average each remaining fibre with one broken neighbour is then carrying the stress $\sigma(i) = \sigma(N/(N-i) + 1/4)$. Between two consecutive breakings, the local stresses do not change. This yields for the time $\tau(i)$ between events i and $(i+1)$: $W/(2N\sigma) = a\sigma(i)\tau(i)$, which is equivalent to

$$\tau(i) = \frac{W}{a2N\sigma^2(1/4 + N/(N-i))} \approx \frac{2W(N-i)}{5aN^2\sigma^2} \tag{3}$$

where we used that $5N \gg i$. The time of the i-th event is then given by:

$$t_i = t_1 + \sum_{j=1}^{i-1} \tau(j) \approx t_1 + \frac{2W}{5N^2 a\sigma^2}(N(i-1) - (i-1)^2/2) \tag{4}$$

where in the last step $N \gg 1$ has been used. It follows for the cumulative damage D which is equal to the event number i:

$$\begin{aligned} D &= i = 1 + N\left(1 - \sqrt{1 - 5a\sigma^2(t_i - t_1)/W}\right) \\ &= 1 + N\left(1 - \sqrt{1 + 4\sigma(1/W - 1)}\sqrt{1 - t_i \frac{5a\sigma^2}{W + 4\sigma(1-W))}}\right) \end{aligned} \tag{5}$$

where we have used $t_1 = 4(1-W)/(5a\sigma)$ (Eq. (2)). If the system aready contained an initial number d of failed fibers, this has to be added to the damage caused by the newly breaking fibres. Thus, the number of available fibres N has to be replaced by $N - d$, and the stress σ has to be replaced by the effective stress $\sigma N/(N - d)$ leading to

$$\begin{aligned} D &= d + i \\ &= d + 1 + (N-d)\left(1 - \sqrt{1 + \frac{4\sigma(1/W-1)}{(1-d/N)}}\sqrt{1 - t_i \frac{5a\sigma^2}{(1-d/N)^2(W + \frac{4\sigma(1-W)}{(1-d/N)})}}\right) \\ &\approx N - (N-d)\sqrt{1 + \frac{4\sigma(1/W-1)}{(1-d/N)}}\sqrt{1 - t_i \frac{5a\sigma^2}{(1-d/N)^2(W + \frac{4\sigma(1-W)}{(1-d/N)})}} \end{aligned} \tag{6}$$

This equation is plotted for a $N = 401 \times 401$ lattice with $a = 1$, $W = 0.1$, $\sigma = 0.01$ and $d = 146$ together with the numerical results in Fig. 1 and labelled (a) on the diagram. The analytical and numerical results are in very good agreement.

3. Breaking of fibres with intact neigbourhood

In this case the smallest critical threshold for the local damage has again the approximate value $c_{th} \approx 1 - W$. However, the local stress is now given by σ and the onset time t'_1 is given by $1 - W = a\sigma t'_1$ which is equivalent to

$$t'_1 = (1-W)/(a\sigma). \tag{7}$$

On average all N fibres have equal distributed thresholds between $1 - W$ and $1 + W$. Thus the average distance between two consecutive thresholds is $2W/N$. After the breaking of i fibres, on average each fibre carries a stress $\sigma'(i) = \sigma N/(N - i)$, i.e. the stress which was carried by N fibres at the very beginning, is now carried by $(N-i)$ fibres. Between two consecutive breakings, the local stresses do not change. This yields for the time $\tau'(i)$ between events i and $(i + 1)$: $2W/N = a\sigma'(i)\tau'(i)$, which is equivalent to

$$\tau'(i) = 2W(N-i)/(N^2 a\sigma). \tag{8}$$

The time of the i-th event is given by:

$$\begin{aligned} t_i &= t'_1 + \sum_{j=1}^{i-1} \tau(j) = t'_1 + \frac{2W}{N^2 a\sigma}(N(i-1) - (i-1)i/2) \\ &\approx t'_1 + \frac{2W}{N^2 a\sigma}(N(i-1) - (i-1)^2/2) \end{aligned} \tag{9}$$

where in the last line $N \gg 1$ has been used. It follows for the cumulative damage D' which is equal to the event number i:

$$D' = i = 1 + N\left(1 - \sqrt{1 - a\sigma(t_i - t'_1)/W}\right) = 1 + N\left(1 - \sqrt{1/W}\sqrt{1 - a\sigma t_i}\right) \tag{10}$$

where we have used $t'_1 = (1 - W)/(a\sigma)$ (Eq. (7)). If the system aready contained an initial number d' of failed fibers, this has again to be added to the damage caused by the newly breaking fibres. Thus, N has to be replaced by $N - d'$, and the stress σ has to be substituted by the effective stress $\sigma N/(N - d')$ leading to

$$\begin{aligned} D' &= d' + i = d' + 1 + (N - d')\left(1 - \sqrt{\frac{1}{W}}\sqrt{1 - \frac{t_i a\sigma}{(1 - d'/N)}}\right) \\ &\approx N - (N - d')\sqrt{\frac{1}{W}}\sqrt{1 - \frac{t_i a\sigma}{(1 - d'/N)}}. \end{aligned} \tag{11}$$

This equation is also plotted in Fig. 1 (labelled (b)) with $d' = 2150$ and N, a, W and σ as above. Again, the analytical and numerical results are in very good agreement.

4. Discussion

At early times the nontrivial behaviour of the time dependent cumulative damage in a FBM including time dependent ageing can be described analytically by considering a sequence of different dominating mechanisms acting in the system. Explicitly, we have derived analytical equations for the two cases of (i) failure of fibres with one damaged neighbour and (ii) failure of fibres with intact neighbourhood. In a similar manner, earlier jumps in the observed damage can be attributed to the case of two damaged neighbours, etc. We note that the above calculations and approximations are only valid for large systems and small applied stresses when the initial damage is limited and damage accumulation due to cascades or other correlated failure triggered by local stress redistribution can be neglected. In these situations our analytical equations are in very good agreement with the numerical simulations. We have not yet determined the times where correlated damage becomes dominant.

Acknowledgments

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