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BGL CONFERENCES: A BRIEF HISTORY

L. L. Jenkovszky

Bogolyubov Institute for Theoretical Physics, Nat. Ac. Sc. of Ukraine
Kiev-143, 03680 UKRAINE and
RMKI (KFKI), POB 49, Budapest 114, H-1525, HUNGARY

Abstract

The birth and the evolution of the BGL series of conferences are briefly reviewed.

I. Prelude

It started in 1997. The idea of the conference, that later gave start to the series, came during a conversation with late N.A.Chernikov in the train Dubna - Moscow (2 hours of journey). Nikolai Alexandrovich Chernikov was a prominent theorist working at the Bogolyubov Laboratory of Theoretical Physics of the Joint Institute for Nuclear Research in Dubna. He was an outstanding expert in general relativity, geometry and quantum field theory. With his wife Natalia, they maintained close relations and scientific collaboration with the Lobachevskij Kazan State University. Nikolai Ivanovich Lobachevsky, born in Nizhni Novgorod, dedicated most of his life to the Kazan University, where he started his studies and his work, later becoming its rector. Both towns lie on the splendid Volga river, heart of Russia, away from Europe's cross-roads. The character of the people, their mentality and their behavior bears much in common with this unique environment.

Speaking about Lobachevsky's life, full of drama, Chernikov said

- You, Hungarians, have an equally great man in your history! His name is János Bolyai, and his life was as tragic as that of our co-patriot. We should remember of them together!

So we decided to call for a conference under the title "Non-Euclidean geometry in modern physics and mathematics" or, in short, BGL, after the names of Bolyai, Gauss and Lobachevsky, where Russians and Hungarians would join their efforts to remember the heritage of their great ancestors. The name of Friedrich Gauss is usually cited among the creators of the new geometry, and we are looking forward for a wider German involvement in future BGL conferences. The ordering in the abbreviation is purely alphabetic. Although there are different opinions about the priority of the discovery of the non-Euclidean geometry, we avoided any preference at this point. The title of the subsequent conferences slightly varied, the last proposal being: "Non-Euclidean geometry and modern physics", but the symbolic abbreviation (BGL) remaining unchanged. The topics of the BGL were settled as: history (from Euclides to the present times), mathematics and physics, the accents depending on the interests of the organizers.

It should be stressed that, apart from its strict physics and geometric content, the conference has also a special "human" or "cultural" aspect in bringing together traditions of the classical science and the spirit of the Old Continent, different from the so-called globalization. BGL is also a bridge between East and West in this changing world. The number of the participants is stable, varying around 50. The first conference gave start to a series of biannial meetings at varying places of Europe.

Let us recall briefly the history of 5 previous BGL conferences, comprising now a period of more that 10 years - a "quasi jubilee".

II. Ungvár-Uzhgorod, Transcarpatia (1997)

The venue of the first BGL was chosen to be in Ungvár, Transcarpatia (now Uzhgorod, Ukraine), where I was born and I have studied at the local university. Transcarpatia is bordering with several countries, located between the Western (Hungarian and German) and Eastern (Slavic) cultural environments and influence, symbolically linking the heritage of Bolyai, Gauss and Lobachevsky and their followers. The local Institute of Electron Physics (IEP) of the Ukrainian Academy of Sciences kindly provided hospitality for the first BGL conference. A bust of N.I. Lobachevsky, by the known Ukrainian sculptor V. Fedichev (Kiev) was inaugurated at the opening, and was donated to the IEP.

The director of the IEP, member of the Ukrainian Academy and foreign member of the Hungarian Academy of Sciences Otto Spenik with the scientific secretary of the Institute Zoltán Tarics consolidated the local organizing committee providing excellent working conditions for the Conference as well as excursions with conference dinners at the villages Nagy Dobrony and Péterfalva. It was, perhaps, for the first time since the end of the 2nd World War that physicists and mathematicians from the neighboring Transcarpatia and Transylvania, separated by less than 100 km (and a border!), could meet and discuss the common cultural heritage. Russia was represented by two great physicists – N.I. Chernikov and A.A. Tyapkin - both from Dubna.

Of crucial importance for the first and subsequent BGL conferences was the support from the Hungarian Academy of Sciences and its member Professor István Lovas, who remains a central figure in the organization of all subsequent BGL meetings. The proceedings of the first BGL conference were published in [1].

III. Nyíregyháza, Eastern Hungary (1999)

The venue of the 2nd BGL meeting, thanks to the efforts of Arpád Szabó, former director of the Hungarian lyceum in Ungvár (Uzhgorod), was the Nyíregyháza Pedagogical Institute, where A. Szabó moved in the meantime. The Institute provided all the necessary facilities (conference hall, lodging and meals at low prices), enabling wide participation at the conference - both from East (Romania, Ukraine, Bielorussia, Russia) and from the West [A. de Alfaro (Torino), M. Tonin (Padova), H. Terazawa (Tokyo), L. Csernai (Bergen) and many others]. For the first time Transylvania, homeland of János Bolyai, was represented by its leading experts on the subject, including Samu Benkő and Tibor Toró (history of science). Participant was also the outstanding, world-wide recognized expert of the Bolyai heritage, Elemér Kiss from Marosvásárhely, where János Bolyai spent most of his life. E. Kiss became an expert on Bolyai's manuscripts and wrote a book on the studies of these manuscripts (being difficult to read!) where, apart from the new geometry, Bolyai's contribution to the number theory is also presented. The book, besides the two Hungarian editions, was translated and printed also in English and is now a bibliographic rarity. Two great men, followers of Lobachevsky and Bolyai, namely N.A, Chernikov and E. Kiss, met during

BGL-2 in Nyíregyháza - for the first and, alas!, the last time. After heavy and long straggle against their disease, both died of cancer (in 2006).

The second BGL meeting in Nyíregyháza reaffirmed the universal and humanistic spirit of the BGL conferences. The social program included an excursion to the famous Tokaj wine yards. The proceedings of the BGL conference were published, due to the invaluable efforts of Prof. I. Lovas, in two issues of the Acta Physica Hungarica [2].

IV. Marosvásárhely – Targu Mures (Transylvania) (2002)

2002 was the year of the widely celebrated 200-th anniversary of János Bolyai. In particular, the Hungarian Academy of Sciences organized a large Bolyai-conference in Budapest in August. We decided to join the celebrations by organizing BGL-3 in September 2002, after an "irregular", 3-years interval. János Bolyai was born in Kolozsvár (Klausenburg, Cluj Napoca), but he lived with his father and died in Marosvásárhely, leaving there more than 20,000 pages of mathematical manuscripts, that can now be found in the Bolyai-Teleki library.

Vice-Mayor of Marosvásárhely Sándor Csegzi, together with academician István Lovas from Budapest and Debrecen were the principal organizers of the BGL-3 conference. The Hungarian Sapientia University of Transylvania, together with the Town Council as well as the Hungarian Academy of Sciences supported the conference. Most of the participants came from Romania, Hungary, Ukraine, Russia and Bielorussia, but there were also participants from far away countries like Japan. The atmosphere of the conference was dominated by the mystical presence of Bolyais – father and son. We visited memorial places of the family, including the cemetery. A more relaxed excursion was organized to neighboring villages, populated by Székelys, "Hungarian cossaks", whose unofficial capital is Marosvásárhely. The Proceedings of BGL-3 are published in [3].

V.Nizhni Novgorod (Russia) (2004)

From Central Europe, BGL moved to North-East, to Russia. In 2004 the Lobachevsky Nizhni Novgorod University was the host of the 4-th conference (see: http://www.unn.ru/bgl4/). It was organized by Prof. F.

Polotovskiy and his staff, supported by the Rector of the University, prof. Strongin.

We enjoyed the cordial Russian hospitality and profited from the high-level presentations, especially those in mathematical physics, the field in which Russia has always a large number of interesting results. The participation of a considerable number of Hungarians at the conference at Russia's heartland, in spite of the barriers imposed by visas, high travel costs and prejudices from mass media, was a proof of the viability and continuity of cultural links between East and West and of the mutual respect for common values represented by the BGL heritage. During the site-seeing, the participants became acquainted with the memorial places of N.I.Lobachevsky. In a boat trip along Volga, the legendary town of N. Novgorod with its majestic Kremlin has opened its splendor. The proceedings of the BGL-4 conference [4] contain a collection of high-level papers in various fields of mathematics and theoretical physics, as well on the history of science.

VI.Minsk (Bielorussia) (2006)

Bielorussia, in spite of its relatively modest dimensions, has a community of physicists and mathematicians, grouped in Minsk and elsewhere. Professor Yury Kurochkin, who participated in most of the previous BGL conferences, is a known expert in geometry and theoretical physics. With his assistant, mathematician Victor Red'kov from the Institute of Physics of the Bielorussian Academy of Sciences, they led the organizing committee of the 5-th BGL conference, held in the fall of 2006, in a resort, outside the city of Minsk (http://dragon.bas-net.by/bgl5/). Similar to the previous conference in N. Novgorod, the hosts provided reasonable low-cost accommodation and food, and excellent, high-level scientific presentations. The program was dominated by contributions form Bielorussia and neighboring countries. An enjoyable excursion to the city of Minsk was organized. A big volume of the Proceedings was published shortly after BGL-5 [5].

VII.Future

This year the Conference returned to Central Europe, Debrecen, heartland of Hungary. During the discussion concluding BGL-6, we heard that:

1. The biennial series should be continued. Several options for the next conference site were mentioned, among them were Kolozsvár (Cluj-Napoca) and Trieste. The optimal title seems to be: "Non-Euclidean geometry in modern physics".

2. A wider German participation, including the organization of a future BGL conference in Germany, is highly welcome.

3. The scope of the conference is right and it should be continued; physics and mathematics should be present in a balanced way, with some history of science, arts etc. added. Ultimately, János Bolyai was a polyhistor, to use this "modern" term. He was an accomplished polyglot, speaking nine foreign languages, including Chinese and Tibetan. He played violin and was a skilled fencer. F. Gauss was learning Russian (to read Pushkin or Lobachevsky?). Their life and heritage are inspiring!

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A NOTE TO K-TORSE-FORMING VECTOR FIELDS ON COMPACT MANIFOLDS WITH COMPLEX STRUCTURE

M. Chodorová, J. Mikeš

Department of Algebra and geometry, Faculty of Science, Palacky University
Olomouc, Czech Republic

Abstract

Certain properties of torse-forming, concircular and convergent vector fields on manifols with affine connection are studied. Connections of manifols in which such vector fields exist are found. Moreover, examples of the mentioned manifols in case they are compact and metrizable are presented.

I. Introduction

Concircular and torse-forming vector fields were introduced by K. Yano [16] in 1944 and their properties in Riemannian spaces have been studied by various mathematicians. Their generalizations are Kählerian torse-forming vector fields (shortly *K-torse-forming*) which were introduced by Yamaguchi [14]. Many authors, for example [2, 10], investigated Kählerian torse-forming vector fields which we call K-concircular vector fields.

Special types of these vector fields (covariantly constant, recurrent, convergent, concircular) have been studied earlier. Riemannian spaces, on which these fields exist, have a specific form of a metric, namely they are warped product spaces, see for example [6, 7, 8, 12, 13, 15].

The vector fields have been studied mostly in Riemannian spaces. Their definitions, as it is shown, depend first of all on an affine connection and basically not on a metric, see [13].

In this paper we introduced local and global conditions of an existence of the studied vector fields on manifolds A_n with torsion-free affine connections and the conditions of setting the metric in A_n. Actually it is a continuation of our previous paper, see [11].

II. K-torse-forming vector fields

First we note definitions and some properties of torse-forming vector fields, via them we define recurrent, convergent and concircular vector fields, see [8].

Definition 1. *A vector field* $\boldsymbol{\xi}$ *on a manifold* A_n *with an affine connection* ∇ *is called torse-forming, if the condition* $\nabla_X \boldsymbol{\xi} = \rho \cdot X + a(X) \cdot \boldsymbol{\xi}$ *holds for any vector field* X *from* $\mathcal{X}(A_n)$*,* ρ *is a function on* A_n*,* a *is a linear form on* A_n*.*

A torse-forming vector field $\boldsymbol{\xi}$ *is called*

- *recurrent, if* $\rho \equiv 0$,
- *concircular, if the form* a *is gradient (or locally gradient), i.e. there exists*
  (locally) a function $\varphi(x)$ *such that* $a = d\varphi(x) = \partial_i \varphi(x)\, dx^i$,
- *convergent, if* $\boldsymbol{\xi}$ *is concircular and* $\rho(x) = \mathrm{const} \cdot \mathrm{e}^{\varphi(x)}$.

Let A_n be an n-dimensional manifold with affine connection ∇ (shortly – *space with affine connection* ∇), on which an affinor structure F is defined (i.e. F is a tensor field of type $\binom{1}{1}$ on A_n), we can define more generalized vector fields.

Definition 2. *A vector field* $\boldsymbol{\xi}$ *is called* K*-torse-forming if*

$$\nabla_X \boldsymbol{\xi} = \rho \cdot X + \sigma \cdot FX + a(X) \cdot \boldsymbol{\xi} + b(X) \cdot F\boldsymbol{\xi}, \qquad \forall X \in TV_n, \tag{1}$$

where ρ*,* σ *are some function, and* a*,* b *are linear forms on* A_n*.*

In local coordinates x it is

$$\xi^h_{,i} = \rho\,\delta^h_i + \sigma\,F^h_i + a_i\xi^h + b_i F^h_\alpha \xi^\alpha,$$

where ξ^h, F^h_i, a_i, b_i are components of $\boldsymbol{\xi}$, F, a, b, and " , " denote the covariant derivative.

These vector fields are studied on Kählerian, eventually on Hermitian, spaces from many others aspects, see for example S. Yamaguchi [3, 14], K.R. Esenov [2], J. Mikeš, G.A. Starko [10], see [7].

It is easy to prove an integral curve ℓ: $x = x(t)$ of a K-torse-forming vector field $\boldsymbol{\xi}$ is F-*planar*, because its tangent vector $\mathrm{d}x/\mathrm{d}t = \boldsymbol{\xi}$ satisfies a following condition ([4, 7, 9])

$$\nabla_\xi \boldsymbol{\xi} = \varrho_1(t)\,\boldsymbol{\xi} + \varrho_2(t)\,F\boldsymbol{\xi},$$

where ϱ_1, ϱ_2 are functions of a parameter t.

An existence of K-torse-forming vector fields on spaces with affine connection has two aspects – *local* and *global*. These aspects were studied for torse-forming and concircular vector fields in [11].

The fundamental question is an existence of spaces A_n, on which mentioned vector fields exist; for example, such global vector fields live on compact spaces.

III. Local existence of K-torse-forming vector fields on A_n

III.1

At first we construct all affine connections on spaces A_n (locally) on which K-torse-forming vector fields exist.

The finding of all spaces A_n with affine connection ∇, on which these fields are defined, is easy from a locally aspect. It is known, that a chart (x, U) exists on manifolds for non vanishing vector field $\boldsymbol{\xi}$ and it holds:

$$\xi^h(x) = \delta^h_1, \quad \forall x \in U.$$

We note $\xi^h_{,i} \equiv \partial_i \xi^h + \xi^\alpha \Gamma^h_{\alpha i}$, where ξ^h and Γ^h_{ij} are components of a vector field $\boldsymbol{\xi}$ and of an affine connection ∇ on spaces A_n. We get the following expression Γ^h_{ij} of affine connection ∇ on spaces A_n on which K-torse-forming vector spaces are defined if we substitute this to the equations (1):

$$\Gamma^h_{1i}(x) = \rho(x)\delta^h_i + \sigma(x)F^h_i(x) + a_i(x)\delta^h_1 + b_i(x)F^h_1(x), \tag{2}$$

where $\rho(x)$, $\sigma(x)$, $a_i(x)$, $b_i(x)$ are some functions defined on U, $F^h_i(x)$ are components of a structure F on U; the other components $\Gamma^h_{ij}(x)$ are arbitrary functions defined on U.

In general case the components (2) can define a connection ∇ with torsion. If $\Gamma^h_{ij} = \Gamma^h_{ji}$ then this connection ∇ is torsion-free.

An analysis of these formulas it follows that a set of manifolds A_n on which mentioned vector fields live is very broad. It is possible to verify that the majority of manifolds A_n are not metrizable, i.e. there does not exist a metric g, for which a connection on A_n is not a Levi-Civita connection of g.

The affinor structure F is arbitrary. Evidently, in the event, if F is complex or almost complex structure, in general case space A_n is not Kählerian or Hermitian space.

III.2

It is well-known [7] a *Kählerian space* is a Riemannian space on which there are defined metric g and complex structure F satisfying

$$F^2 = -Id, \qquad g(X, FY) + g(FX, Y) = 0, \qquad \nabla F = 0,$$

for all tangent vectors X, Y.

In paper by J. Mikeš and G.A. Starko [10] there was introduced a metric of a Kählerian space and in this space there exists a K-torse-forming (or K-*concircular*) vector field. In the canonical coordinate system x this metric has a following expression:

$$g_{ab} = g_{a+mb+m} = \partial_{ab}G + \partial_{a+mb+m}G, \quad g_{ab+m} = g_{a+mb} = \partial_{ab+m}G - \partial_{a+mb}G,$$

where $G = G(x^1 + s(x^2, x^3, \dots, x^m, x^{m+2}, x^{m+3}, \dots, x^{m+m}))$, $G', G'' \neq 0$, $G, s \in C^3$, are functions of mentioned arguments, $a, b = 1, \dots, m$, $m = n/2$, the structure F is canonical, i.e. $F_b^{a+m} = -F_{b+m}^a = \delta_b^a$, $F_b^a = F_{b+m}^{a+m} = 0$, and $\partial_i = \partial/\partial x^i$. In this coordinate system a K-torse-forming vector field is expressed: $\boldsymbol{\xi} = \partial_1$.

IV. Global existence of K-torse-forming vector fields on compact A_n

IV.1

We introduce an example of a space with affine connection which is made on n-dimensional torus.

Let $A^n = S^1 \times S^1 \times \dots \times S^1$, and $x^1, x^2, \dots, x^n$, be the corresponding angles on the circles. We have global vector fields $X_1 = \partial_1, X_2 = \partial_2, \dots, X_n = \partial_n$.

We define the affine connection ∇ through its actions on these vector fields, as follow:

$$\nabla_{X_i} X_1 = \rho(x) X_i + \sigma(x) F X_i + a(X_i) X_1 + b(X) F X_1,$$

and for the others $\qquad \nabla_{X_i} X_j = \sum_{k=1}^{n} \omega_{ij}^k(x)\, X_k, \quad j \neq 1,$

where $\rho, \sigma, \omega_{ij}^k$ are functions and a, b are linear forms on A_n, and F is an affinor structure on A_n.

Evidently, *the space A_n is compact, and $\boldsymbol{\xi} \equiv X_1$ is a K-torse-forming vector field.*

The structure F on even-dimensional A_n, for which the following conditions hold

$$F X_a = X_{a+m}, \quad F X_{a+m} = -X_a, \qquad \forall a = 1, \dots, m, \quad 2m = n,$$

is a globally *complex structure*. It is known, the following expression $F^2 = -Id$ holds for this structure.

IV.2

We introduce an example of a compact space with torsion-free affine connection and covariantly constant complex structure which is made on n-dimensional torus.

Let $A^n = S^1 \times S^1 \times \cdots \times S^1$, and $x^1, x^2, \ldots, x^n$, $n = 2m$, be corresponding angles in circles. Global vector fields are defined: $X_1 = \partial_1, X_2 = \partial_2, \ldots, X_n = \partial_n$.

We define complex structure F and affine connection ∇, by actions of these vector fields:

$$FX_a = X_{a+m}, \quad FX_{a+m} = -X_a, \qquad \forall a = 1, \ldots, m, \tag{3}$$

$$\nabla_{X_i} X_j = \sum_{k=1}^{n} \omega_{ij}^k(x) X_k, \tag{4}$$

where ω_{ij}^k $(= \omega_{ji}^k)$ are functions on A_n.

It has been assumed that the functions ω_{ij}^k satisfies

$$\begin{aligned} \omega_{ab}^c &= \omega_{a\,b+m}^{c+m} = -\omega_{a+m\,b+m}^c, \\ \omega_{a+m\,b+m}^{c+m} &= \omega_{a\,b+m}^c = -\omega_{a\,b}^{c+m}, \quad a, b, c = 1, 2, \ldots, m. \end{aligned} \tag{5}$$

Then we prove that the structure F is covariantly constant, i.e. $\nabla F = 0$, see [5].

Moreover, if

$$\omega_{a1}^c = \omega_{a\,1+m}^{c+m} = \omega_{a+m\,1}^{c+m} = -\omega_{a+m\,1+m}^c = \psi_a \delta_1^c + \psi_1 \delta_a^c,$$

$$\omega_{a+m\,1+m}^{c+m} = \omega_{a\,1+m}^c = -\omega_{a\,1}^{c+m} = \psi_{a+m} \delta_1^c - \psi_{1+m} \delta_a^c,$$

where ψ_i are functions on A_n, then the vector field $\boldsymbol{\xi} = X_1$ is K-torse-forming.

Lemma 1. *There exists a compact manifold A_n with torsion-free affine connection and globally defined covariantly constant complex structure and K-torse-forming vector field.*

Furthermore we suppose that

$$\omega^1_{11} = \omega^{1+m}_{1\,1+m} = \omega^{1+m}_{1+m\,1} = -\omega^1_{1+m\,1+m} = 1,$$

and the other components of ω are zero. The formulas (4) and (3) define a torsion-free affine connection ∇ and a covariantly constant affine structure F on A_n, respective. A vector field $\boldsymbol{\xi} = X_1$ is K-torse-forming.

Locally this connection ∇ is calculated in terms of a metric $g = \mathrm{diag}(g_{11}, g_{22}, \dots, g_{nn})$, where

$$g_{11} = g_{1+m\,1+m} = \exp(2x^1),\ g_{aa} = g_{a+m\,a+m} = 1,\ a = 2, \dots, m,\ 2m = n.$$

Evidently, this metric locally generates a Kählerian space with the structure F.

In other hand, the constructed space A_n is not globally metrizable.

From this follows that $\nabla_{\boldsymbol{\xi}}\boldsymbol{\xi} = \boldsymbol{\xi}$, and for the lenght $|\boldsymbol{\xi}| = \sqrt{g(\boldsymbol{\xi}, \boldsymbol{\xi})}$, we have $\nabla_{\boldsymbol{\xi}}|\boldsymbol{\xi}| = |\boldsymbol{\xi}|$. Because, A_n is compact, this case does not exist.

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ON GEODESIC MAPPINGS OF AFFINE CONNECTION MANIFOLDS

J. Mikeš[1], I. Hinterleitner[2], V. Kiosak[3]

[1] Fac. of Sci., Palacky University, Olomouc, Czech Repulic
[2] FSI Technical University, Brno, Czech Republic
[3] Schiller University, Jena, Germany

Abstract

In this paper we prove that all affine connection manifolds are locally projectively equivalent to some space with equiaffine connection (equiaffine manifold). We found a system of linear equations which determine all (pseudo-) Riemannian spaces admitting geodesic mappings onto an a-priori defined space with affine connection.

I. Levi-Civita equations of geodesic mappings

As well known, a *geodesic mapping* is a diffeomorphism which preserves geodesic curves, see for example [1]-[20], etc.

Beltrami [1] in 1865 began to study geodesic mappings onto Euclidean spaces. Levi-Civita [7] obtained fundamental equations of geodesic mappings between Riemannian spaces. H. Weyl [19] defined geodesic mappings between affine connection manifolds. He showed that the Levi-Civita equations are valid in this case, too.

These results were first formulated only locally. Many times it was found that the Levi-Civita equations hold also globally (“ in whole”), see [8].

Let A_n and $\bar{A}_n$ be n-dimensional affine connection manifolds with connections ∇ and $\bar{\nabla}$, respectively. We suppose that there exists a diffeomorphism $f\colon A_n \to \bar{A}_n$. Because it is very well known [3, 8, 12, 16, 19] that an affine connection manifold is projectively equivalent to a manifold with symmetric affine connection, we suppose that the connections ∇ and $\bar{\nabla}$ are symmetric affine connections.

If $U \subset A_n$ is a coordinate neighborhood with coordinates $x = (x^1, \ldots, x^n)$, we suppose that the points $M \in U$ and $\bar{M} = f(M) \in f(U)$ have identical coordinates x. These coordinates x are called *common coordinates* of the mapping f.

A diffeomorphism $f\colon A_n \to \bar{A}_n$ is a geodesic mapping if and only if the following *Levi-Civita equation* holds:

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \delta^h_i \psi_j(x) + \delta^h_j \psi_i(x), \tag{1}$$

where Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are components of ∇ and $\bar{\nabla}$, respectively, ψ_i is covector, δ^h_i is the Kronecker symbol.

A diffeomorphism f from the manifold A_n onto the (pseudo-) Riemannian manifold $\bar{V}_n$ is a geodesic mapping if and only if the following *Levi-Civita equation* holds:

$$\bar{g}_{ij,k} = 2\,\psi_k\,\bar{g}_{ij} + \psi_i\,\bar{g}_{jk} + \psi_j\,\bar{g}_{ik}, \tag{2}$$

where $\bar{g}_{ij}(x)$ are components of the metric tensor $\bar{g}$ of $\bar{V}_n$, " , " denotes the covariant derivative with respect to the connection ∇ on A_n.

With the aid of these equations many problems of geodesic mappings of Riemannian manifolds and affine connection manifolds were solved.

Levi-Civita [7], see [3]-[20], obtained these fundamental equations for geodesic mappings between Riemannian manifolds. The above Levi-Civita equations hold equally for Riemannian and for pseudo-Riemannian manifolds.

In the following we suppose that (see [13, 14, 16]):

Riemannian manifold $\equiv$ Riemannian and pseudo-Riemannian manifold.

II. Geodesic mappings and equiaffine connection manifolds

As we have already said, affine connection manifolds are projectively equivalent to some spaces with symmetric affine connection. Note that a symmetric affine connection ∇ is called *equiaffine* if the Ricci tensor of A_n is symmetric [12, 16].

It is known [12, 16] that the manifold A_n is *equiaffine* (this means A_n has an equiaffine connection), if and only if on a coordinate neighborhood U there exists a function $f(x)$ so that $\Gamma^{\alpha}_{i\alpha}(x) = \frac{\partial f(x)}{\partial x^i}$.

We have the following theorem.

Theorem 1. *An affine connection manifold is locally projectively equivalent to an equiaffine manifold.*

Proof. Let A_n be a manifold with affine connection ∇. We can restrict ourselves to the case that ∇ is symmetric. We suppose that a coordinate neighborhood $U \in A_n$ is mapped geodesically on $\bar{A}_n$ under the assumption of the validity of the Levi-Civita equations (1).

We construct a covector $\psi_i(x)$ in the following way:

$$\psi_i(x) = -\frac{1}{n+1}\Gamma^{\alpha}_{i\alpha}(x). \tag{3}$$

From (1) and (3) follows

$$\bar{\Gamma}^{\alpha}_{i\alpha}(x) = 0. \tag{4}$$

Formulae (3) and (4) hold only in the distinguished coordinate system x, because $\bar{\Gamma}^{\alpha}_{i\alpha}(x)$ is not a covector. Condition (4) is equivalent to the symmetry of the Ricci tensor of $\bar{A}_n$ and the equiaffinity of $\bar{A}_n$. This property is not dependent of coordinates.

III. Mikeš-Berezovski equations of geodesic mappings from equiaffine manifolds onto Riemannian manifolds

Sinyukov started from the following problem: *find all Riemannian manifolds $\bar{V}_n$ which admit geodesic mappings onto an a priori defined Riemannian manifolds V_n*, see [8, 16].

This means we must find all metric tensors $\bar{g}$, which are solutions of the Levi-Civita equations (1) and (2). These equations are non-linear with respect to the components of the metric tensor $\bar{g}$ and for their solution no standard methods exist. Sinyukov (see [8, 16]) for this problem obtained a set of linear equations of Cauchy type.

Mikeš and Berezovski started from the generalized problem: *find all Riemannian manifolds $\bar{V}_n$ which admit geodesic mappings onto an a priori defined affine connection manifold A_n*, see [8, 9].

Theorem 2 (Mikeš, Berezovski [8, 9])**.** *The equiaffine manifold A_n admits a geodesic mapping onto a Riemannian manifold $\bar{V}_n$, if and only if the complete set of linear differential equations of Cauchy type in the covariant derivatives in A_n*

$$\begin{array}{llcl} (a) & a^{ij}_{\ \ ,k} & = & \lambda^i \, \delta^j_k \; + \; \lambda^j \, \delta^i_k; \\ (b) & n \, \lambda^i_{\ ,j} & = & \mu \, \delta^i_j + a^{i\alpha} R_{\alpha j} - a^{\alpha\beta} R^i_{\alpha\beta j}; \\ (c) & (n-1)\, \mu_{,i} & = & 2(n+1)\lambda^\alpha R_{\alpha i} + a^{\alpha\beta}(2R_{\alpha i,\beta} - R_{\alpha\beta,i}) \end{array} \qquad (5)$$

has a solution with respect to the unknown symmetric regular tensor a^{ij}, the vector λ^i, and the function μ. The solutions of this system and (1) *are related by the equality*

$$a^{ij} = \exp(2\psi)\, \bar{g}^{ij}; \quad \lambda^i = -\exp(2\psi)\, \bar{g}^{i\alpha}\psi_\alpha, \qquad (6)$$

where ψ_i is a gradient vector of the function ψ, $\bar{g}^{ij}$ are components of the dual tensor of the metric tensor of $\bar{V}_n$.

Here R^h_{ijk} and $R_{ij} = R^\alpha_{ij\alpha}$ are components of the Riemannian and Ricci tensors of A_n, the comma " , " denotes the covariant derivative in A_n.

The first formula (5) gives the necessary and sufficient condition for the existence of a geodesic mapping: $A_n \to \bar{V}_n$. This mapping is nontrivial if and only if $\lambda_i \not\equiv 0$.

In this case, the set of equations (5) is linear and its solution is reduced to the investigation of the integrability conditions and their differential prolongations, which are a set of algebraic (homogeneous with respect to the unknown tensors a^{ij}, λ^i, and μ) equations with coefficients from A_n (i.e. coefficients formed from objects defined on A_n). Thus, in principle, we can solve the following problem, if the given equiaffine manifold A_n admits geodesic mappings onto the Riemannian manifold $\bar{V}_n$ and if the choice of this mapping is arbitrary.

This system has not more than only one solution for initial conditions in the point $x_\circ$:

$$a^{ij}(x_\circ) = \overset{\circ}{a}{}^{ij}, \quad \lambda^i(x_\circ) = \overset{\circ}{\lambda}{}^i \quad \mu(x_\circ) = \overset{\circ}{\mu}.$$

The general solution of Eqs. (5) depends on a finite number of substantial parameters $r \leq N_0 \equiv \dfrac{(n+1)(n+2)}{2}$. The number r is called the *degree of mobility* of A_n with respect to geodesic mappings onto Riemannian manifolds. From here it follows that the set of manifolds $\bar{V}_n$ onto which A_n admits geodesic mappings, depends on a set of parameters of cardinality not exceeding r.

The *degree of mobility* of A_n with respect to geodesic mappings onto $\bar{V}_n$ was investigated in [8, 9]. In this work, it was shown that the maximum value $r = \frac{(n+1)(n+2)}{2}$ is achieved only in projective-Euclidean manifolds, and for nonprojective-Euclidean A_n $(n > 2)$ it is true that $r = \frac{n(n+2)}{2} + 2$.

By a detailed analysis it can be shown that Theorem 2 holds for $A_n \in C^2$, i.e. for all the components $\Gamma^h_{ij}(x) \in C^2$ of the affine connection ∇.

IV. Linear equations of geodesic mappings from affine connection manifolds onto Riemannian manifolds

In the paper [9] by Mikeš and Berezovski (see [8]) a system of equations of Cauchy type for geodesic mappings from an affine connection manifold A_n onto a Riemannian manifold $\bar{V}_n$ was found. These equations are non linear.

From Theorem 1 and the equations (5) the existence of linear equations follows also for this general case.

Assume that A_n admits a geodesic mapping onto the equiaffine manifold $\tilde{A}_n$ under the condition

$$\tilde{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) - \frac{1}{n+1}\,(\delta^h_i \Gamma^\alpha_{j\alpha}(x) - \delta^h_j \Gamma^\alpha_{i\alpha}(x)). \tag{7}$$

and $\tilde{A}_n$ admits a geodesic mapping onto the Riemannian manifold $\bar{V}_n$ with the metric $\bar{g}$.

The first formula of (5) holds

$$a^{ij}{}_{|k} \equiv \partial_k a^{ij} + a^{\alpha j}\tilde{\Gamma}^i_{\alpha k} + a^{\alpha i}\tilde{\Gamma}^j_{\alpha k} = \lambda^i \delta^j_k + \lambda^j \delta^i_k \tag{8}$$

where " $|$ " is the covariant derivative on $\tilde{A}_n$.

By insertion of (7) into (8) we find an equation for a geodesic mapping from A_n onto $\bar{V}_n$ in the following form

$$a^{ij}{}_{,k} \equiv \partial_k a^{ij} + a^{\alpha j}\Gamma^i_{\alpha k} + a^{\alpha i}\Gamma^j_{\alpha k} = \frac{2}{n+1} a^{ij}\, \Gamma^\alpha_{\alpha k} + \delta^i_k \Lambda^j + \delta^j_k \Lambda^i, \tag{9}$$

where

$$\Lambda^i = \lambda^i + \frac{1}{n+1} a^{i\beta}\Gamma^\alpha_{\beta\alpha}.$$

Equations (9) are linear with respect to the unknown functions $a^{ij}(x)$ and $\lambda^i(x)$. These equations hold in the chosen coordinate system x. Their solutions are tensors $a^{ij}(x)$ and $\lambda^i(x)$, which do not depend on the choice of coordinates.

For each solution of the equations (9), with the aid of formulae (6), a metric $\bar{g}$ of the Riemannian manifold $\bar{V}_n$ can be found.

Theorem 3. *The manifold A_n admits a geodesic mapping onto a Riemannian manifold $\bar{V}_n$ if and only if there exists a solution of* (9) *with respect to the unknown functions $a^{ij}(x)$* ($\det \|a^{ij}(x)\| \not\equiv 0$) *and $\lambda^i(x)$. The metric $\bar{g}$ of $\bar{V}_n$ satisfies the conditions* (6).

The geodesic mappings of projective Euclidean manifolds are studied in detail in the monographs [4, 12, 16].

By a detailed analysis of the integrability conditions of equations (9) and their first differential prolongations it can be shown that in coordinate neighborhoods, where A_n is not projectively Euclidean, the vector λ^i can be expressed in the form

$$\lambda^i = a^{\alpha\beta}(x)\, G^i_{\alpha\beta}(x), \tag{10}$$

where $G^i_{\alpha\beta}(x)$ is determined by objects of the affine connection of A_n.

Then the equations (9) form a closed linear system of Cauchy type with respect to the unknown functions $a^{ij}(x)$.

Proof. Now we can prove formula (10).

Asume that A_n maps geodesicaly on a Riemannian manifold $\bar{V}_n$. Than in each coordinate neighbourhood $U \subset A_n$ the equations (9) have a solution.

We restrict ourselves to the case, that in the coordinate neighbourhood $U(x)$ A_n is not projectively flat, i.e. the Weyl tensor of projective curvature is non vanishing, $W^h_{ijk}(x) \neq 0$.

For the coordinate neighbourhood $U(x)$ we further construct a series of geodesicaly mapping manifolds

$$A_n \to \tilde{A}_n \to \bar{V}_n,$$

where $\tilde{A}_n$ is an equiaffine manifold.

Equations (9), valid in A_n, have in $\tilde{A}_n$ the form (8). The integrability condition of (8) can be written in the form

$$a^{\alpha(i}\tilde{R}^{j)}_{\alpha kl} = \lambda^{(i}_{|l}\delta^{j)}_k - \lambda^{(i}_{|k}\delta^{j)}_l\,. \tag{11}$$

where $\tilde{R}^h_{ijk}$ is the Riemannian tensor of $\tilde{A}_n$.

Because in an equiaffine manifold $\tilde{A}_n$ the Weyl tensor of projective curvature has the following form

$$\tilde{W}^h_{ijk} = \tilde{R}^h_{ijk} - \frac{1}{n-1}(\delta^h_k\tilde{R}_{ij} - \delta^h_j\tilde{R}_{ik}),$$

where $\tilde{R}_{ij}$ is the Ricci tensor of $\tilde{A}_n$, and this tensor is an invariant of geodesic mappings, i.e. $\tilde{W}^h_{ijk} = W^h_{ijk}$, formula (11) can be written in the following form

$$a^{\alpha(i}W^{j)}_{\alpha kl} = \Lambda^{(i}_l \delta^{j)}_k - \Lambda^{(i}_k \delta^{j)}_l \,,$$

where Λ^i_l is a tensor and W^h_{ijk} is the Weyl tensor of projective curvature of A_n.

The covariant derivative of the last formule with respect to x^m in $A_n$is

$$a^{\alpha(i}_{,m}W^{j)}_{\alpha kl} + a^{\alpha(i}W^{j)}_{\alpha kl,m} = \Lambda^{(i}_{l,m}\delta^{j)}_k - \Lambda^{(i}_{k,m}\delta^{j)}_l \,.$$

After insertion of (9) acquires the following principial form

$$\lambda^{(i}W^{j)}_{mkl} + a^{\alpha\beta}T^{ij}_{\alpha\beta klm} = \Lambda^{(i}_{lm}\delta^{j)}_k - \Lambda^{(i}_{km}\delta^{j)}_l + L^{(i}_{k\,l}\delta^{j)}_m \,. \tag{12}$$

where $T^{ij}_{\alpha\beta klm}$ is an object determined by the connection ∇ of A_n and Λ^h_{lm}, L^h_{lm} are objects.

In [11] it was proved that for $n > 2$ when $W \neq 0$ there exists a coordinate system x in which $W^1_{223} \neq 0$. One by one we insert into (12):

$$\begin{aligned}
&i = 1, \ldots, n,\ j = 1,\ m = k = 2,\ l = 3;\\
&i = j = k = 1,\ l = 3,\ m = 2;\\
&i = j = m = 1,\ l = 3,\ k = 2;\\
&i = j = k = 1,\ l = m = 2
\end{aligned}$$

and we can see that (10) holds.

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SOME QUESTIONS OF FINSLER- AND DISTANCE-GEOMETRIES

L. Tamássy

Institute of Mathematics, H-4010, Debrecen, P. O. Box 12, Hungary

1. This is a Conference on Non-Euclidean Geometry and its Applications, called also Bolyai-Gauss-Lobachevsky Conf. So I feel it pertinent to pay respect in a few words to these scientific luminaries, and at the same time give a reason for speaking here on Finsler geometry. I would like to start with a few facts of the history of mathematics.

Proofs first appeared in mathematics after the penetration of the ideas of Greek philosophy, only about in the fifth century B.C. With this the ever steepest development started in the history of mathematics. Only one and a half century later Euclid was able to write his famous book, the Elements, in which he could deduce every theorem from a few axioms. Among these the last one was the well-known parallel axiom. Nevertheless this axiom raised problems. Many asked whether this is a real axiom, or else it can be proven. The problem turned out to be very hard. It refused any attack through two thousand years. Finally it was solved by Bolyai and Lobachevsky (Bolyai found it first, and published later, Lobachevsky found it later, and published it first). The answer affirmed the rank of the parallel axiom. The problem was very difficult indeed, but the answer did not alter the geometry at all. Everything remained as before. It seemed that the solution required a really big effort, but yielded a modest result. Nevertheless they proved their result by constructing a *new* geometry, and this was of the utmost significance. The importance of this construction can be compared to the turn from the geocentric world concept to the heliocentric one. The possibility, the existence of another geometry was unconceivable for nearly all of the mathematicians of the time. It became properly recognized, it gained its right to its proper place only slowly. Gauss, who also was interested in the problem, and who had nice partial results, was the first who understood and accepted the idea of Lobachevsky and Bolyai. However, for some only partially acceptable reasons, he did not want to propagate the new geometry. Yet in spite of all difficulties the new geometry spread out. After the first highly difficult steps new and new geometries appeared. In 1854 Riemann presented the basic ideas of "Riemannian geometry". This happened at his habilitation lecture under the chairmanship of the old Gauss (next year Gauss died). That the new ideas spread but slowly is excellently shown by the fact that Riemann's ideas were published first only after his death (1866) in the volume of his Collected Works (1892), and Riemann geometry became developed in the XX-th century only. Today we have a number of geometries, most of them with successful applications in physics, among them also Finsler geometry. Thus Finsler geometry is the son, or at least the grandson of Lobachevsky and Bolyai, and on this right I dare to speak today on some problems of Finsler geometry.

2. First *a* few introductory words on Finsler and distance geometries. We have two types of metrical differential geometries: i/ those built on the arc length of curves, ii/ those built on the distance of two points. Since these are differential geometries, in both cases everything must be differentiable (of class C^∞).

We consider first the geometries which are built on the arc length. Let $\gamma(t) \subset M$, $a \leq t \leq b$, $\dot\gamma(t) \neq 0$ be a curve of a manifold M. Then

$$s(t) := \int_a^b \|\dot\gamma(t)\|\, dt \tag{1}$$

is a quite natural and generally used definition for the arc length. Clearly the tangent space T_pM, $p \in M$ must be a normed vector space. What kind of norm ? We put three simple and very natural requirements on s, which uniquely determine the type of the norm $\|\,.\,\|_p$, $p \in M$. These requirements are the following:

A) $s > 0$

B) s is independent of any orientation-preserving parameter transformation.

C) $\| \,.\, \|_p$ satisfies the triangle inequality.

It is clear that

A): $s > 0 \Longleftrightarrow \|y\| > 0, \;\; y \in T_pM, \;\; y \neq 0.$

B): Let $t = t(\tau), \;\; \tau = \tau(t), \;\; \tau(a) = \alpha, \;\; \tau(b) = \beta$ be a parameter transformation. This preserves the orientation if $\frac{dt}{d\tau} > 0$. Then s is independent of the parameter transformation $t = t(\tau)$ iff

$$\int_a^b \|\dot{\gamma}(t)\| \, dt = \int_\alpha^\beta \|\gamma'(\tau)\| \, d\tau = \int_a^b \left\| \dot{\gamma}(t) \frac{dt}{d\tau} \right\| \frac{dt}{d\tau} \, dt. \tag{2}$$

Since $\dot{\gamma}(t)$ can be any vector of T_pM and $\frac{dt}{d\tau}$ may be any positive number, B) is equivalent to

$$\|\lambda\gamma\|_p = \lambda \|y\|_p, \quad y \in T_pM, \qquad \lambda \in R^+,$$

where R^+ denotes the positive reals.

Finally C) says that

$$\|y_1 + y_2\|_p < \|y_1\|_p + \|y_2\|_p, \quad y_1, y_2 \in T_pM, \quad y_1 \neq \mu y_2, \quad \mu \in R.$$

Thus the requirements A), B), C) are equivalent to the following properties of the norm:

I) $\|y\|_p > 0$ if $y \neq 0$

II) $\|\lambda y\|_p = \lambda \; \|y\|_p, \;\; \lambda \in R^+$

III) $\|y_1 + y_2\|_p < \|y_1\|_p + \|y_2\|_p, \;\; y_1 \neq \mu y_2, \;\; y, y_1, y_2 \in T_pM, \;\; \mu \in R.$

I), II), III) characterize the Banach norm. Thus a geometry built on the arc length satisfies the very natural requirements A), B), C) iff the norm applied in (1) is a Banach norm, which depends on the point $p \in M$.

It is a difference only in notation, if we introduce the function

$$\mathcal{F}(p, y) := \|y\|_p.$$

Then we define a Finsler space $F^n = (M, \mathcal{F})$ over a manifold M by giving a fundamental (or metric) function $\mathcal{F}(p, y)$ with the properties I), II), III), and we define the arc length of a curve $\gamma(t) \subset M$ by $s := \int_a^b \mathcal{F}(\gamma(t), \dot{\gamma}(t))\, dt$ (see [1]). Thus Finsler geometry is the most general geometry satisfying the very natural requirements A), B), C). If the Banach norm reduces to a Euclidean norm, then we obtain a Riemann geometry. It is easy to see that C) or III) is equivalent to the convexity of the indicatrix (see (8)), and this convexity is equivalent to the property that in the simplest cases (Euclidean or Minkowski geometry) geodesics are straight lines. This is another geometric expression of the requirement C) or III). Finsler geometry and its numerous simple special cases offer many possibilities for physical applications. This is so, because Finsler geometry has much more free parameters or functions, than Riemannian geometry.

The other type of metrical differential geometries are distance spaces $D^n = (M, \varrho)$ (see [2]). A distance space over M is given by a distance function $\varrho : M \times M \to R^+$ ordering to any ordered pair (p, q) of points a non-negative real. This function can be symmetric: $\alpha)$ $\varrho(p, q) = \varrho(q, p)$, and it can satisfy the triangle inequality: $\beta)$ $\varrho(p, q) + \varrho(q, r) \geq \varrho(p, r)$. If both $\alpha)$ and $\beta)$ are satisfied, then D^n is called metric. If $\alpha)$ may fail, then D^n is called quasi-metric. In what follows we consider quasi-metric distance spaces. Metric distance spaces are contained as a special case.

3. What is the relation between distance spaces $D^n = (M, \varrho)$ and Finsler spaces $F^n = (M, \mathcal{F})$ over the same manifold M ? Any Finsler metric determines a distance function ϱ^F by

$$\varrho^F(p, q) := \inf_{\Gamma} s(\gamma(p, q)), \tag{3}$$

where Γ means the collection of the curves from p to q, and $s(\gamma(p, q))$ means their arc length. Then ϱ^F is non-negative and satisfies the triangle inequality. Thus

$$\mathcal{F} \Longrightarrow \varrho^F \quad \text{and} \quad F^n = (M, \mathcal{F}) \Longrightarrow D^n = (M, \varrho^F).$$

Is this relation invertible ? Does also $\mathcal{F}$ determine ϱ^F? Yes, namely

$$\lim_{t \to 0^+} \frac{d}{dt} \varrho^F(p_0, g(t)) = \mathcal{F}(p_0, y_0), \tag{4}$$

where $g(t)$, $0 \leq t < \varepsilon$ is a geodesic of F^n emanating from $p_0 = g(0)$, and y_0 is its (one sided) tangent at $p_0 : y_0 = \dot{g}(0)$. (4) is a famous result of H. Busemann and W. Mayer [3] (see also [1], p. 158). It can be proved easily. If ε is small, then $g(t)$ is a "short geodesic", which minimizes the arc length between $g(0) = p_0$ and $g(t)$. Hence, by (3)

$$\varrho^F(p_0, g(t)) = s(p_0, g(t)) = \int_0^t \mathcal{F}(g(\tau), g'(\tau))d\tau. \tag{4'}$$

By (one sided) differentiation we obtain (4). This shows that

$$\mathcal{F} \Longrightarrow \varrho^F \Longrightarrow \mathcal{F} \Longrightarrow \varrho^F \ldots,$$

i.e. the relation between $\{\mathcal{F}\}$ and $\{\varrho^F\}$ is $1:1$. We remark that $\frac{d}{dt}\varrho^F(p_0, g(t))$ equals the directional derivative of ϱ^F:

$$\lim_{t \to 0^+} \frac{d}{dt}\varrho^F(p_0, g(t)) = \frac{d}{dt}\Big|_{p_0, y_0} \varrho^F(p_0, q), \quad q \in M.$$

Now, this relation does not contain the geodesic $g(t)$, so starting with a distance function ϱ of a distance space $D^n = (M, \varrho)$

$$\mathcal{F}(p_0, y_0) := \frac{d}{dt}\Big|_{p_0, y_0} \varrho(p_0, q) \tag{5}$$

defines a function $\mathcal{F}(p, y)$. One can show that this $\mathcal{F}$ satisfies A), B) C). Hence this $\mathcal{F}$ is a fundamental function of a Finsler space $F^n = (M, \mathcal{F})$. Thus we obtain

$$\varrho \Longrightarrow F \Longrightarrow \varrho^F. \tag{6}$$

But *is this* ϱ^F *of* (6) *equal to the starting* ϱ? We show that in general *it is not*. This can be shown by an example, where $\varrho \Longrightarrow \mathcal{F} \Longrightarrow \varrho^F \neq \varrho$. First let M be 1-dimensional: $M = R^1$ with canonical coordinates x. Let us define

$$\varrho(x_0, x)) := \ln(|x - x_0| + 1). \tag{7}$$

One can check that this ϱ is non-negative, symmetric, and satisfies the triangle inequality. So it is a distance function of a metric space $D^1 = (R^1(x), \varrho)$. By (5) it determines a Finsler metric $\mathcal{F}(p, y)$, which turns out to be absolutely homogeneous, and independent of x_0. Therefore the constructed

$F^1 = (R^1, F)$ is a Minkowski space with symmetric indicatrix, and because of $n = 1$ it is a Euclidean space. Hence

$$\varrho^F(x_1, x_2) = |x_1 - x_2|.$$

By the integral mean theorem

$$\varrho(x_1, x_2) = \int_{x_1}^{x_2} \varrho'(x_1, x)\,dx = (x_2 - x_1)\varrho'(x_1, x)\big|_{x_0}$$

$$x_1 < x_2, \quad x_0 \in (x_1, x_2),$$

$$\varrho'(x_1, x) = \frac{d}{dx}\varrho(x_1, x).$$

The derivative of ϱ (given by (5)) is strictly decreasing on $x > x_1$, and $\varrho'(x_0, x)|_{x_0} = 1$. Thus $\varrho'(x_1, x_0) < 1$ and hence

$$\varrho(x_1, x_2) = (x_2 - x_1)\varrho'(x_1, x)|_{x_0} < |x_1 - x_2| = \varrho^F(x_1, x_2),$$

showing that $\varrho \neq \varrho^F$.

This example can be extended to $M = R^n$ $(n > 1)$. In this case we define the function $z = \varrho(0, x)$, $(x, z) \in R^{n+1}$ by the rotation of $z = \ln(|x| + 1)$ (see (7)) around the z axis, and we define $\varrho(x_0, x) := \varrho(0, x - x_0)$. Also other examples over $M \neq R^n$ can be constructed.

These show that there are many distance spaces $D^n = (M, \varrho)$ such that ϱ determines by (5) the same Finsler space and the same ϱ^F, but only for one of these is $\varrho^F = \varrho$ in (6).

For which distance spaces $D^n = (M, \varrho)$ *does* $\varrho \Longrightarrow \mathcal{F} \Longrightarrow \varrho^F = \varrho$ *hold?* The answer needs a little more preparation. In [4] *we gave necessary and sufficient conditions for this.* The basic idea is the following. In an $F^n = (M, \mathcal{F})$ along a short geodesic $g(t)$, $0 \leqq t \leqq T$ by (4') we obtain

$$\varrho^F(p, g(t)) = \varrho^F(p, g(t_1)) + \varrho^F(g(t_1), g(t)),$$

where $p \in g(\tau)$, $0 \leq \tau < t_1 < t < T$. From this

$$\left[\frac{d}{dt}\varrho^F(p, g(t))\right]_{t_1} = \left[\frac{d}{dt}\varrho^F(g(t_1), g(t))\right]_{t_1^+}$$

for every $p \in g(\tau)$, $0 \leq \tau < t_1$. This means that the functions $\varrho^F(p, g(t))$, which measure the distance from the different $p \in g(\tau)$ to $g(t)$ have the same derivative at t_1, and their graphs have parallel tangents at t_1. A curve of D^n with similar property is called "parallelity curve". In the proof we show that the existence of such a parallelity curve between any pair of points of a distance space D^n is necessary and sufficient for $\varrho = \varrho^F$.

4. We show still another interesting global result of Finsler geometry. For the sake of simplicity we restrict ourselves to a two-dimensional absolutely homogeneous Finsler space $F^2=(M,\mathcal{F})$. The indicatrix $\mathcal{I}(p_0)$ of an $F^n=(M,\mathcal{F})$ is a hypersurface of the tangent space defined by

$$\mathcal{I}(p_0) := \{y \in T_{p_0}M \mid \mathcal{F}(p_0, y) = 1\}. \tag{8}$$

$\mathcal{I}(p_0)$ is a generalization of the unit sphere $\mathcal{S}^{n-1}$ of the Euclidean space E^n. If $\varphi : M \to M$ is a motion of F^n, then the linear mapping $d\varphi$ takes $\mathcal{I}(p)$ into $\mathcal{I}(\varphi(p))$. This means that $\mathcal{I}(p)$ and $\mathcal{I}(\varphi(p))$ must be affine equivalent. Now suppose that

a) p_1 and p_2 are such points of F^2 that $\mathcal{I}(p_1)$ and $\mathcal{I}(p_2)$ are not affine equivalent to any other $\mathcal{I}(p)$ of F^2

b) let F^2 be geodesically complete, i.e. there exists a geodesic between any pair of points of F^2

c) let the injectivity radii $\iota(p_1)$ and $\iota(p_2)$ be such that $\iota(p_1) + \iota(p_2) \leq \varrho(p_1, p_2)$. In consequence of this there exist geodesic circles $\mathcal{S}_{p_1}(r_1)$ with radius $r_1 \leq \iota(p_1)$, centered at p_1, and $\mathcal{S}_{p_2}(r_2)$ with radius $r_2 < \iota(p_2)$, centered at p_2

d) there exists in F^2 a 1-parameter continous group of motions $\varphi_t \neq \mathrm{id}$.

We claim that *under these conditions there exists a diffeomorphism* $\Psi : F^2 \to \varphi \subset E^3$ *where* φ *is a revolution surface, and moreover* Ψ *is an isometry for the meridians and parallels of* φ *[5].*

We sketch the proof. Since F^2 is geodesic complete there exists a geodesic g between p_1 and p_2. Let $q_0 \in g$ be such that $\varrho(p_1, q_0) \leq \iota(p_1)$ and $\varrho(p_2, q_0) \leq$

$\iota(p_2)$. Then there exist two geodesic circles $\mathcal{S}_{p_1}(k_1)$, $k_1 = \varrho(p_1, q_))$, and $\mathcal{S}_{p_2}(k_2)$, $k_2 = \varrho(p_2, q_))$, through q_0. p_1 and p_2 are fix points of φ_t, for $\mathcal{I}(p_1)$ and $\mathcal{I}(p_2)$ are not affine equivalent to any other $\mathcal{I}(p)$. Hence q_0 can move only on $\mathcal{S}_{p_1}(k_1)$ by any motion φ_t. Furthermore it is easy to see that q_0 cannot be a fix point of φ_t, for in this case φ_t would be the identity. Thus q_0 can be taken into any point of $\mathcal{S}_{p_1}(k_1)$ by an appropriate φ_t. Nevertheless the same is true also for $\mathcal{S}_{p_2}(k_2)$. Therefore $\mathcal{S}_{p_1}(k_1) = \mathcal{S}_{p_2}(k_2)$. Finally we claim that $M = B_{p_1}(k_1) \cup B_{p_2}(k_2)$, where $B_{p_1}(k_1)$ is the closed disk of M bounded by $\mathcal{S}_{p_1}(k_1)$, and similarly $B_{p_2}(k_2)$. Namely if q ($\neq p_1$) is an arbitrary point of M, then there exists a geodesic g^* through q, and emanating from p_1. g^* intersects $\mathcal{S}_{p_1}(k_1)$ and $\mathcal{S}_{p_2}(k_2)$ perpendicularly at a point q^* and runs further in $\mathcal{S}_{p_2}(k_2)$ to p_2, and then further to a common point q^{**} of $\mathcal{S}_{p_1}(k_1)$ and $\mathcal{S}_{p_2}(k_2)$. Therefore q must lie on g^* between p_1 and p_2, and thus in $B_{p_1}(k_1) \cup B_{p_2}(k_2)$. Since both $B_{p_1}(k_1)$ and $B_{p_2}(k_2)$ are diffeormorphic to a hemisphere of $\mathcal{S}^2 \subset E^3, B_{p_1}(k_1) \cup B_{p_2}(k_2) = M$ is diffeomorphic to the unit sphere $\mathcal{S}^2$ or to a revolution surface φ of E^3.

We can show a little more. Let ψ be a diffeomorphism from F^2 to a revolution surface φ of E^3. We can choose φ in such a way that the images of the geodesic circles $\mathcal{S}_{p_1}(r)$, $r \leq k_1$ and $\mathcal{S}_{p_2}(r)$, $r \leq k_2$ are parallels of φ, and the images of the geodesics g_α, $\alpha \in \mathcal{A}$ from p_1 to p_2 are meridians of φ. The radii of these parallels can be so that the Euclidean arc length of the parallels is equal to the Finsler arc length of the corresponding geodesic circle. Let $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ be two geodesic circles from the family $\{\mathcal{S}_{p_1}(r),\ r \leq k_1;\ \mathcal{S}_{p_2}(r),\ r < k_2\}$. $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ cut out a segment s_α from each g_α. The Finsler arc length of s_α is independent of α, for any two different s_α are taken into each other by a motion φ_t. Therefore φ can be chosen such that the Finsler arc length of the geodesics g_α equals the Euclidean arc length of the corresponding meridian of φ. Thus $\psi : F^2 \to \varphi \subset E^3$ satisfies the announced properties.

There are several similar results. L. Green [6], M. Berger and J. L. Kazdan [7] and C. T. Yang [8] showed that if in a Riemannian space $V^n = (M, g)$ the cut locus of any point $p \in M$ consists of a single other point (these manifolds are called "Wiedersehen" manifolds) then this V^n is isometric to the Euclidean sphere $\mathcal{S}^n$ (see also J. L. Kazdan [9]]). In our case the cut locus of p_1 is p_2. So this property is fulfilled for one pair of points only, but we have another severe condition, the existence of the motions φ_t. In fact our result, which concerns a Finsler space is weaker. F^2 is only diffeomorphic to φ, and isometry holds only on the parallels and on the meridians of φ.

In case of an n-dimensional Finsler space F^n the points p_1 and p_2 of the assumption a) must be replaced by n points $p_1, p_2, \ldots, p_n$ in general position, having similar properties as p_1, p_2, or in d) φ_t must be replaced by an $n-1$ parameter continuous group of motions $\varphi_{t_1 \ldots t_{n-1}}$. In these cases the proof is a little longer.

There are many interesting results on isometries of Finsler spaces, such as in [10] by S. Deng and Z Hou, or in [11] by L. Kozma and P. Radu. I mention here extra a result of S. Deng [12]. He showed that if the connected sets $V_i \subset M$ of a Finsler space $F^n = (M, \mathcal{F})$ consist of the zeros of a Killing vector field ξ, then V_i are totally geodesic submanifolds of F^n. This can be related to the affine equivalence of the indicatrices considered in our talk. If in a Finsler space $F^n = (M, \mathcal{F})$ no indicatrix $\mathcal{I}(p)$, $p \in M$ is affine equivalent to an $\mathcal{I}(p_1)$, then p_1 must be a zero of any Killing vector field ξ. Thus if $p_\beta \in M$, $\beta \in \mathcal{A}$ are such points as p_1, and $\{p_\beta\} = V$ is a submanifold, then this V is totally geodesic in F^n. Also the other results of Deng hold on such submanifolds $V = \{p_\beta\}$.

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METRIZATION OF LINEAR CONNECTIONS, HOLONOMY GROUPS AND HOLONOMY ALGEBRAS

A. Vanžurová

Fac. Sci., Dept. Algebra and Geometry, Palacký University,
Tomkova 40, 779 00 Olomouc, Czech Republic,
alena.vanzurova@upol.cz

Abstract

Metrization problem means: given a manifold endowed with a (symmetric) linear connection, decide whether the connection arises from some metric tensor as its Levi-Civita connection. Compatibility conditions for a metric are given by a system of ordinary differential equations, and the classical approach is to analyze the system of integrability conditions. Let us present more geometric solution procedure using parallel transport, emphasize the role of holonomy groups and holonomy algebras. The problem is of some interest in itself (e.g. [6]; S.B. Edgar, J. Math. Phys. **33**, 3716 (1992); [7], [8], [9]); we propose one application: for a particular type of second order system of ODEs, coefficients give rise to a connection; provided it is metrizable, components of the compatible metric play the role of variational multipliers for the Inverse Problem and yield (one of) the Lagrangian(s).

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I. Metrization as a kind of inverse problem

Let (M, g) be a pseudo-Riemannian manifold*. A (pseudo-)Riemannian metric g on M determines uniquely a canonical linear (= affine) connection ∇ on M, called the *Levi-Civita* connection of (M, g), the characterizing properties of which are $T \equiv 0$ ($\Gamma^i_{jk} = \Gamma^i_{kj}$) and $\nabla g = 0$. The inverse problem called Metrization Problem (MP) is: *Given a manifold* (M, ∇) *with a linear symmetric connection, is there a metric on* M *the Levi-Civita connection of which is just* ∇*?* It belongs probably to the oldest and in a way difficult problems of classical differential geometry. A similar problem can be posed for a linear connection in an arbitrary vector bundle, particularly in the tangent bundle of a manifold, or in Finsler spaces ([1], L. Tamassy, Balkan J. of Geom. and Appl. **1** (1996) etc). Related problems are: *If there are more such metrics, how much may they differ from each other?* (The answer is closely related to the concept of the de Rham - Wu decomposition.) *Given* (M, g), *find all metrics with the same Levi-Civita connection.* All multiples rg, $r \in R$, have this property, and if there are no others we speak about *uniqueness* of the metric. But if the manifold admits the de Rham - Wu decomposition there might be the so-called *alternative metrics*, [8]. MP is related to the theory of geodesic mappings†. An equivalent formulation of MP is: given (M, ∇), find all possible geodesic mappings $f{:}M \to \bar{M}$ of (M, ∇) onto (pseudo-)Riemannian manifolds $(\bar{M}, \bar{g})$. Hence tensor methods developed‡ in the theory of geodesic mappings may be used. Our problem is also related to the Calculus of Variations. The so-called Inverse Problem (IP) of the calculus of variations (still open) is: if a system $\ddot{x}^i = f^i(t, x^k, \dot{x}^k)$ of SODEs§ is given, decide whether it represents Euler-Lagrange equations of some Lagrangian, i.e. find¶ Lagrangian functions $L(t, x^k, \dot{x}^k)$ and a multiplier matrix $g_{ij}(t, x^k, \dot{x}^k)$ such that $g_{ij}(\ddot{x}^i - f^i) \equiv \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i}$. Complete so-

*M is an n-dimensional manifold of a "sufficiently high" class of differentiability, and g is a non-degenerate metric, that is, a symmetric type $(0, 2)$ tensor field on M with local componens g_{ij} satisfying $\det(g_{ij}) \neq 0$, not necessarily positive definite.

†Recall that if we are given two manifolds with linear connection (M, ∇) and $(\bar{M}, \bar{\nabla})$, respectively, a (smooth or C^r-differentiable, $r \geq 1$) bijection $f : M \to \bar{M}$ is called a *geodesic mapping* if any (canonically parametrized) geodesic γ of (M, ∇) is mapped onto an unparametrized (= arbitrarily parametrized) geodesic $\bar{\gamma}$ of $(\bar{M}, \bar{\nabla})$.

‡N.S. Sinyukov, *Geodesic Mappings of Riemannian Spaces*, Moscow, 1979.

§second order differential equations; $i, k = 1, \ldots, n$.

¶sufficiently differentiable

lution is known only for $n = 2$ (J. Douglass, 1941). Hence MP can be viewed as a particular case[||] of IP, where $f^i = -\Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k$, when the multipliers are time- and velocities-independent; then kinetic energy $L = \frac{1}{2}g_{ij}(x)\dot{x}^i\dot{x}^j$ (comming from MP) is one of the Lagrangians solving IP (there might more general ones). Also the metric uniqueness problem was related to the general inverse problem of Lagrangian dynamics[**]. During the time, various methods used for solving MP (eventually under some constraints) were suggested and developed by various authors, from most straightforward ones, [2], based on analysis of integrability conditions for ODEs, to more sophisticated ones, [6], [4], [5], [11] and the references therein, based either on tensor methods, or employing parallel transport induced by connection, or their combinations, etc. Low-dimensional cases have been discussed separately e.g. in [7], [10] ($n = 2$), [9] ($n = 3$). Positive definite metrics for a symmetric connection with regular curvature were constructed in [4]. Existence of positive definite metrics for an analytic connection on an analytic manifold is decided in [5] by means of an algorithm based on properties of de Rham decomposition and the fact that in the analytic case, the Lie "holonomy" algebra is spanned by the curvature tensor and its covariant derivatives (Ambrose-Singer Theorem); in the affirmative case, all compatible Riemannian metrics are effectively constructed, [11].

II. Classical approach - differential equations

The (pseudo-)Riemannian connection of (M, g) is uniquely determined by zero torsion and the condition $\nabla g = 0$, telling in an elegant way that the parallel transport induced by the connection should preserve the scalar pruduct. If (M, ∇) is given, $\nabla g = 0$ represents the system of ODEs for unknowns g_{ij}

$$\frac{\partial g_{ij}}{\partial x^k} - g_{sj}\Gamma^s_{ik} - g_{is}\Gamma^s_{jk} = 0 \tag{1}$$

which should be discussed under the assumption $\det(g_{ij}) \neq 0$. In simple cases, the system (1) can be solved directly. Note that a solution of (1) might not be a metric, if non-degeneracy condition $\det g_{ij} \neq 0$ is not satisfied; the

[||] In fact, provided $\det g_{ij} \neq 0$, the system $\ddot{x}^i + \Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k = 0$ is equivalent to the system $g_{mi}(\ddot{x}^i + \Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k) = 0$, $i, m = 1, \ldots, n$.

[**] G. Marmo, C. Rubano, G. Thompson, Class. Quantum Grav.**7**, 2155 (1990).

solution depends on n. The integrability conditions for (1) (necessary for metrizability) read, in coordinate-free form,

$$g(\nabla^r R(X,Y;Z_1;\ldots;Z_r)(Z),W) + g(Z,\nabla^r R(X,Y;Z_1;\ldots;Z_r)(W)) = 0$$

for all $X,Y,Z,W,Z_1,\ldots,Z_r \in \mathcal{X}(M)$, $0 \le r < \infty$, which is in fact an infinite homogeneous system of linear equations in g_{ij} with coefficients being functions in $\Gamma's$ and their partial derivatives. For a metrizable connection, the above linear conditions must stabilize for some positive integer $r = N$, i.e. from the $(N+1)^{\text{th}}$ stage, the conditions must be algebraic consequences of the previous ones. We get no conditions for a flat connection ($R = 0$), which is always metrizable (the system has $\frac{1}{2}n(n+1)$-parametrical solution). For $n = 2$, $R \ne 0$ (regular), the answer is relatively easy: *Local necessary and sufficient condition for a nowhere-flat symmetric connection ∇ on M_2 be metrizable are: the Ricci tensor* Ric *of ∇ should be non-degenerate (*$\det R_{ij} \ne 0$*), symmetric (*$R_{ij} = R_{ji}$*) and recurrent,* $\nabla\mathrm{Ric} = \omega \otimes \mathrm{Ric}$ *where ω is some one-form. If ω is exact, $\omega = df$ for some function f, then compatible metrics exist globally, one of the representants being $g = e^{-f}\,\mathrm{Ric}$, the other differ upto a scalar multiple (i.e. g is "unique"). If M_2 is simply connected, a compatible g exists globally.*

If both kinds of points, flat ($R(x) = 0$) and non-flat ($R(x) \ne 0$) are present, we may expect complications. For any $n \ge 2$, there exist non-metrizable n-dimensional affine spaces. A classical algorithm, which brings a prescriptive solution (not in a closed form), was known already since 1920', [2]. The result can be formulated as follows (a free paraphrase):

Theorem 4. *A manifold (M,∇) with a linear connection ∇ and the curvature tensor R is metrizable if and only if the homogeneous equations*

$$g_{sj}R^s_{ik\ell} + g_{is}R^s_{jk\ell} = 0 \tag{2}$$

are "algebraically consistent" (more precisely, the system has at least one-dimensional solution space of non-degenerate metrics), and any solution of (2) satisfies

$$g_{sj}R^s_{ik\ell;m} + g_{is}R^s_{jk\ell;m} = 0, \quad i,j,k,\ell,m \in \{1,\ldots,n\}. \tag{3}$$

The proof is instructive, yields a method for finding compatible metrics using several steps from the proof (and can be implemented to a computer). Suppose that (2) is solvable, and that any solution of (2) satisfies

(3). Choose a basis $\langle G^{(1)}, \dots, G^{(p)} \rangle$ of the solution space. Any solution g can be now written in the form $g = \sum_{\alpha=1}^{p} \varphi^{(\alpha)} G^{(\alpha)}$ with coefficients $\varphi^{(\alpha)}$ which are at most functions of coordinates (x^i) on M. Due to (3), covariant derivatives $G^{(\alpha)}_{sj;m}$ satisfy (2), too, hence $G^{(\alpha)}_{ij;k} = \sum_{\beta=1}^{p} \mu_k^{(\alpha\beta)} G^{(\beta)}_{ij}$. Since second covariant derivatives satisfy the Ricci indentity we get $G^{(\alpha)}_{ij;k\ell} - G^{(\alpha)}_{ij;\ell k} = 0$, and consequently

$$\frac{\partial \mu_k^{(\alpha\beta)}}{\partial x^\ell} - \frac{\partial \mu_\ell^{(\alpha\beta)}}{\partial x^k} + \sum_{\gamma=1}^{p} \left(\mu_k^{(\alpha\gamma)} \mu_\ell^{(\gamma\beta)} - \mu_\ell^{(\alpha\gamma)} \mu_k^{(\gamma\beta)} \right) = 0. \tag{4}$$

If $\nabla g = 0$ should hold, φ's must satisfy

$$\frac{\partial \varphi^{(\alpha)}}{\partial x^k} + \sum_{\beta=1}^{p} \varphi^{(\beta)} \mu_k^{(\alpha\beta)} = 0, \quad \alpha = 1, \dots, p. \tag{5}$$

But according to (4), the system (5) is completely integrable, hence there exist functions $\varphi^{(1)}, \dots, \varphi^{(p)}$ which determine a compatible (pseudo-)Riemannian metric. Let us demonstrate the method presented above on a simple example.

Example 1. The system $\ddot{x} + \dot{x}^2 \cdot x/(x^2+1) = 0$, $\ddot{y} + \dot{y}^2 \cdot y/(y^2+1) = 0$ gives rise to a symmetric linear connection ∇ on R^2 with non-zero components $\Gamma^1_{11} = x/(x^2+1)$, $\Gamma^2_{22} = y/(y^2+1)$; $R \equiv 0$ (the connection is flat hence metrizable). The solution space is a span of independent (global analytic) type $(0,2)$ symmetric tensor fields $G^{(1)} = dx \otimes dx$, $G^{(2)} = dy \otimes dy$, $G^{(3)} = dx \otimes dy + dy \otimes dx$. Their covariant derivatives must be combinations of the generators, $G^{(1)}_{ij;1} = -\frac{2x}{x^2+1} G^{(1)}_{ij}$, $G^{(1)}_{ij;2} = G^{(2)}_{ij;1} = 0$, $G^{(2)}_{ij;1} = -\frac{2y}{y^2+1} G^{(2)}_{ij}$, $G^{(3)}_{ij;1} = -\frac{x}{x^2+1} G^{(3)}_{ij}$, $G^{(3)}_{ij;2} = -\frac{y}{y^2+1} G^{(3)}_{ij}$; we have $\mu_1^{(11)} = -\frac{2x}{x^2+1}$, $\mu_1^{(22)} = -\frac{2y}{y^2+1}$, $\mu_1^{(33)} = -\frac{x}{x^2+1}$, $\mu_2^{(33)} = -\frac{y}{y^2+1}$, zero otherwise. All compatible metrics are $g = \varphi^{(1)} G^{(1)} + \varphi^{(2)} G^{(2)} + \varphi^{(3)} G^{(3)}$ where functions $\varphi' s$ solve (5); $\varphi^{(1)} = -\frac{x}{x^2+1}$ etc. All compatible metrics g are of the form $g = \varphi^{(1)} G^{(1)} + \varphi^{(2)} G^{(2)} + \varphi^{(3)} G^{(3)}$. We get

$$(g_{ij}) = \begin{pmatrix} 2b_2(x^2+1) & b_1\sqrt{x^2+1}\sqrt{y^2+1} \\ b_1\sqrt{x^2+1}\sqrt{y^2+1} & 2b_3(y^2+1) \end{pmatrix}$$

with parameters $b_1, b_2, b_3 \in R$. In tensor notation, $g = 2b_2(x^2+1)\underset{\cdot}{\mathrm{x}} \otimes \underset{\cdot}{\mathrm{x}} + b_1\sqrt{x^2+1}\sqrt{y^2+1}\underset{\cdot}{\mathrm{x}} \otimes \underset{\cdot}{\mathrm{y}} + b_1\sqrt{x^2+1}\sqrt{y^2+1}\underset{\cdot}{\mathrm{y}} \otimes \underset{\cdot}{\mathrm{x}} + 2b_3(y^2+1)\underset{\cdot}{\mathrm{y}} \otimes \underset{\cdot}{\mathrm{y}}$, or classically, $ds^2 = 2b_2(x^2+1)dx^2 + 2b_1\sqrt{x^2+1}\sqrt{y^2+1}dxdy + 2b_3(y^2+1)dy^2$. For admissible Riemannian metrics, b_i should be chosen so that g be positive definite.

III. Geometric approach - Parallel Transport

The “classical" method mentioned above works, but gives little insight into a geometric meaning of the integrability conditions and their consequences for the given connection. To make things more transparent and geometric let us realize what follows. The *holonomy* of (M, ∇) at $x \in M$ around a piecewise-differentiable loop μ (i.e. closed curve with x as starting point as well as endpoint; the class C^1 is sufficient, [3, I, p. 85, Th. 7.2.]; loops are taken with usual composition, [3]) is an automorphism τ_μ of the tangent space T_xM which is given by parallel propagation of vectors along the given loop. Due to properties of the parallel transport along curves ($\tau_{\mu^{-1}} = \tau_\mu^{-1}$, $\tau_\mu \circ \tau_\eta = \tau_{\eta\mu}$), all holonomies at x together with composition form the so-called (full linear) *holonomy group* Hol_x^∇ of (M, ∇) at x, which is a Lie transformation group; using local coordinates about x, it identifies with a subgroup of $GL(n, R)$. Its component of unit is the *restricted holonomy group* Hol_x^0; it is obtained by a similar construction if we take loops homotopic to zero only; $\underline{h}(x) = \underline{\mathrm{Hol}}_x^\nabla$ denotes a common Lie algebra. According to the Ambrose-Singer Theorem, [3], if the connection is smooth (C^∞), the so-called *infinitesimal holonomy algebra* $\underline{h}'(x) \subset \underline{h}(x)$ is a span of the linear maps $\nabla^k R(X, Y; Z_1, \dots, Z_k)$, $X, Y, Z_1, \dots, Z_k$ from T_xM, $0 \le k < \infty$. The above inclusion might be sharp, but in particular cases, the Lie algebras coincide. For a real analytic connection on a real analytic manifold, $\underline{h}'(x) = \underline{h}(x)$ holds, hence $\underline{h}(x)$ can be calculated from the curvature tensor and its covariant derivatives, and Hol_x^0 can be retrieved. If the underlying manifold M is connected, holonomy groups of the connection in different points are isomorphic, $\mathrm{Hol}_x^\nabla \simeq \mathrm{Hol}_y^\nabla$, $x, y \in M$, so let us write Hol^∇. If M is connected, simply connected then Hol^∇ is a connected Lie subgroup of the automorphism transformation group $GL(T_xM)$ of the fibre; hence it is uniquely determined by its Lie algebra $\underline{h} = \underline{\mathrm{Hol}}^\nabla$.

If the connection is metrizable then the parallel transport preserves scalar product, holonomies are isometries in each tangent space; the holonomy group preserves the metric tensor, and identifies with a subgroup of $O(p,q)$, $p+q=n$, according to the signature of g; Hol^0 identifies with a subgroup of the special orthogonal group $SO(p,q)$. The idea of making use of holomomy groups for solution of metrization problem for linear connections was discussed e.g. in [6], [1]. The holonomy group "decides" whether a connection is metrizable or not: obviously, a connection can only be a pseudo-Riemannian connection of a metric g, if the (restricted) holonomy group is a subgroup of the (special) generalized orthogonal group corresponding to the signature. Another formulation: (M_n, ∇) is metrizable if and only if the bundle of all frames is reducible to the orthogonal group $O(p,q)$. In a way, the condition is also sufficient; if Hol_x^0 is a subgroup of the special orthogonal group of the fibre at one point then the compatible metric can be found:

Theorem 5. ([1, Th. 3.1., p. 282], a free paraphrase) *Let (M, ∇) be an affine manifold with M connected. Let there be a point $x_0 \in M$ such that the (restricted) holonomy group is contained in the (special) generalized orthogonal group of T_{x_0}. Then ∇ is metrizable.*

Proof. Fixing a chart around $x_0 \in M$, we may assume that the tangent space $T_{x_0}M$ is isomorphic to $(R^n, \langle , \rangle)$ where $\langle \, , \, \rangle$ denotes the standard scalar product of the corresponding signature. Since M is connected, any point $x \in M$ can be connected with x_0 by a curve in M, and the holonomy groups Hol_x^∇, $\mathrm{Hol}_{x_0}^\nabla$ are isomorphic via parallel transport. We can use parallel propagation to pull the scalar product back.

IV. Riemannian metrics

For Riemannian metrics the following tells that no ambiguity arises in the regular case:

Theorem 6. [4, p. 133] *Let M be a connected manifold with* $\dim M \geq 3$. *Let R be the curvature of (M, g), where g is a Riemannian metric on M, and let the subset D of all regular points of R be dense in M. Then g is determined on D by its curvature tensor R uniquely upto scaling by constants.*

If (M, g) is a (pseudo-)Riemannian manifold with curvature R then at any point $x \in M$, we have a linear map $R_x : \Lambda^2(T_xM) \to \mathrm{End}\,(T_xM)$ such that if $w = \sum_i c_i X_i \wedge Y_i \in \Lambda^2(T_xM)$ then $R_x(w)(Z) = \sum_i c_i R(X_i, Y_i)Z$ for $Z \in T_xM$. Let us generalize properties of the Riemannian curvature R as follows. Let G be a positive definite symmetric bilinear form in T_xM. A linear map $\varrho : \Lambda^2(T_xM) \to \mathrm{End}\,(T_xM)$ will be called a *curvature structure with respect to* G if the following holds: (i) $G(\varrho(X \wedge Y)(Z), W) + G(Z, \varrho(X \wedge Y)W) = 0$; (ii) $G(\varrho(X \wedge Y)Z, W) = G(\varrho(Z \wedge W)X, Y)$ for any $X, Y, Z, W \in T_xM$. In (T_xM, g), $\varrho = R_x$ is a natural example. A linear map $\varrho : \Lambda^2(T_xM) \to \mathrm{End}\,(T_xM)$ will be called *regular* if $\varrho(w) \neq 0$ whenever $w \neq 0$, and *singular* otherwise. Particularly, the subset of all regular points of the Riemannian curvature R of (M, g) is open in M.

Lemma 1. *Let G be a positive definite symmetric bilinear form on T_xM, and ϱ its curvature structure. Then for any G-orthogonal pair $X, Y \in T_xM$, $X \neq 0$, there is a bivector $w \in \Lambda^2(T_xM)$ such that $\varrho(w)X = Y$. If there exists a regular curvature structure ϱ with respect to G then H_ϱ is one-dimensional.*

Theorem 7. *Let (M, ∇) be an affine manifold with a torsion-free linear connection ∇, let the curvature R be regular on M, and let $H^0(M) = \bigcup_{x \in M} H_{R_x}$ be the bundle corresponding to the curvature tensor. Then ∇ is a Riemannian connection of a positive-definite metric g if and only if the following conditions hold:*
(1) $H^0(M)$ is the line bundle (i.e. all fibres are one-dimensional),
(2) the bundle $H^0(M)$ is metric in the Riemannian sense (that is, there is a positive definite symmetric biliear form (on T_xM) in each $H^0(x)$),
(3) any Riemannian metric $h\, M \to H^0(M)$ is recurrent, $\nabla h = \omega \otimes h$, and the 1-form ω is exact on M, i.e. $\omega = df$ for a function f.

Proof. If $h : M \to H^0(M)$ is a Riemannian metric such that $\nabla h = -2df \otimes h$ then we easily check that $g = e^{2f}h$ is a metric compatible with ∇ since $\nabla g = 0$ holds. To prove that the conditions are necessary is a bit more complicated, [4].

As already mentioned, in general we can not calculate the holonomy group from the curvature tensor (and its covariant derivatives), it might be even difficult to find the holonomy group at all, as well as a quadratic form invariant under it. The real analytic case on a connected simply connected manifold is more favourable, [5]. To translate invariance of a symmetric bilinear (quadratic) form relative to holonomy group into the language of holonomy Lie algebra we use the Lemma telling how the assumptions on Hol^{∇} can be reformulated as assumptions on $\underline{h}$:

Lemma 2. *Let (M,∇) be a simply connected smooth manifold with ∇ torsion-free, $x \in M$ a fixed point. Given a symmetric bilinear form G on T_xM then the following holds: G is invariant by Hol^{∇} if and only if*

$$G(AX,Y) + G(X,AY) = 0 \quad \textit{for all } A \in \underline{h}(x),\ X,Y \in T_xM. \tag{6}$$

Proof. We check here that elements of the holonomy algebra satisfy (6). The other implication also holds but the proof is not so trivial. If $A \in \underline{h}(x)$ consider the corresponding one-parameter subgroup $s^A : R \to \mathrm{Hol}^{\nabla}$, $t \mapsto s^A(t)$ uniquely determined by the initial data $s^A(0) = 1$, $(s^A)'(0) := (\frac{d}{dt})_{t=0}\, s^A(t) = A$. Let G be invariant under the holonomy group, $G(\tau X, \tau Y) = G(X,Y)$ for any $\tau \in \mathrm{Hol}^{\nabla}$. Then we get $G(s^A(t)X, s^A(t)Y) = G(X,Y)$ for $X, Y \in T_xM$. Differentiating with respect to t, making use of the formula for scalar product, and considering $t \to 0$ we get (6),

$$G((s^A)'(0)(X), s^A(0)(Y)) + G(s^A(0)(X), (s^A)'(0)(Y)) = 0.$$

The above gives us a quite natural motivation for introducing the vector subspace $H(x) = \{G_x \in S^2(T_x^*M) \,|\, G_x(AX,Y) + G_x(X,AY) = 0,\ A \in \underline{h}(x) \text{ for } X,Y \in T_xM\}$, $x \in M$.

Theorem 8. *Let (M,∇) be connected and let there exist $G_{x_0} \in H(x_0)$ (i.e. G_{x_0} is invariant under Hol^{∇}). Then ∇ is the Levi-Civita connection of a metric on M which has the same signature as G_{x_0}.*

If ∇ is Riemannian (comes from a positive definite metric) then for every $x \in M$, $H(x)$ includes a positive definite form; under additional assumptions, the converse also holds: ([5, Prop. 1], [6]) *Given a connected simply connected (M,∇) and $x \in M$, let there be a positive definite form $G_{x_0} \in H(x)$. Then ∇ is Riemannian.*

It might be difficult to check whether there is a positive definite form in $H(x)$; no direct decision algorithm based on linear algebra only is available. An effective algorithm (deciding Riemannian metrizability in real analytic case) using geometric properties of the Levi-Civita conection and the de Rham decomposition of the tangent space T_xM of a Riemannian manifold (M, g) was developed [5], [11], together with an effective prescription how to construct all compatible Riemannian metrics. Note that for indefinite metrics, the situation is more complicated.

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GROUPS OF BASIC AUTOMORPHISMS OF FOLIATIONS WITH TRANSVERSE RIGID GEOMETRIES

N. I. Zhukova

Department of Mechanics and Mathematics, Nizhny Novgorod State University, 603950, Nizhny Novgorod, Gagarina ave., 23, korp. 6, Russia

Abstract

We introduced the notion of rigid geometry. Foliations (M, F) with transverse rigid geometries were investigated. An invariant $\mathfrak{g}_0$ of (M, F), where $\mathfrak{g}_0$ is a Lie algebra, was constructed. We proved that $\mathfrak{g}_0 = 0$ is a sufficient condition for the unique existence of a Lie group structure in the full basic automorphism group of this foliation. Some estimates of the dimension of this group depending on the transverse geometry were founded. Examples, illustrating the main results, are constructed.

I. Introduction

One of the basic objects associated with a geometric structure on a smooth manifold is its automorphism group. Among the central problems, there is the question whether the automorphism group can be endowed with a (finite-dimensional) Lie group structure [1].

In the theory of foliations with transverse geometries, automorphisms are understood as diffeomorphisms mapping leaves onto leaves and preserving transverse geometries. The group of all automorphisms of a foliation (M, F) with transverse geometry is denoted by $\mathcal{A}(M, F)$. Let $\mathcal{A}_L(M, F)$ be the normal subgroup of $\mathcal{A}(M, F)$ formed by automorphisms mapping each leaf onto itself. The quotient group $\mathcal{A}(M, F)/\mathcal{A}_L(M, F)$ is called the full basic automorphism group and is denoted by $\mathcal{A}_B(M, F)$.

In the investigation of foliations (M, F) with transverse geometry it is naturally to ask the above problem about the existence of a Lie group structure for the full group $\mathcal{A}_B(M, F)$ of basic automorphisms of (M, F).

Leslie [2] was first who solved a similar problem for smooth foliations on compact manifolds. For foliations with complete transversal projectable affine connection this problem was studied by Belko [3].

The leaf space M/F of the foliation is a diffeological space, and the group $\mathcal{A}_B(M, F)$ can be considered as a subgroup of the diffeological Lie group $\mathrm{Diff}(M/F)$. For Lie foliations with dense leaves on a compact manifold, the diffeological Lie groups $\mathrm{Diff}(M/F)$ are computed by Hector and Macias-Virgos [4].

In this work we introduce a notion of a rigid structure. Cartan geometries [1] and rigid geometric structures in the sense of Gromov [5] are rigid structures in our sense. A manifold equipped with a rigid structure is called a rigid geometry.

We investigate foliations admitting rigid geometries as transverse structures and call them by foliations with transverse rigid geometries (TRG). Cartan foliations [6, 7], foliations admitting a transverse systems of differential equations in the sense of Wolak [8] and G-foliations, where G is a Lie group of finite type, are foliations with TRG. In particular, Riemannian [9], pseudo-Riemannian, Lorenz, projective and conformal foliations belong to the class of foliations under investigation. The category of foliations with TRG is denoted by $\mathfrak{F}_{TRG}$. The group $\mathcal{A}_B(M, F)$ is an invariant of (M, F) in the category $\mathfrak{F}_{TRG}$. We always assume that the foliations under consideration are complete and transverse rigid geometries are effective.

We constructed a foliated bundle for a foliation (M, F) with TRG and reduced problems on the automorphism groups and the basic automorphism groups of (M, F) to the analogous problems for e-foliations (Theorems 3 and 6).

For any foliation (M, F) with TRG we defined the structure Lie algebra $\mathfrak{g}_0(M, F)$ and showed that $\mathfrak{g}_0(M, F)$ is an invariant of this foliation in the category $\mathfrak{F}_{TRG}$ (Proposition 3). One of the main results of this work is the theorem asserting that if $\mathfrak{g}_0(M, F)$ is zero, then there exists a unique Lie group structure on $\mathcal{A}_B(M,F)$. We obtained some estimates of the dimensions of these Lie groups depending on the transverse geometries (Theorem 7). We gave different interpretations of holonomy groups of foliations with TRG (Theorem 5) and found some other sufficient conditions for the existence of a Lie group structure on $\mathcal{A}_B(M,F)$ (Theorem 8).

Recall that a foliation is said to be proper if each its leaf is an embedded submanifold of the foliated manifold. In particular, the structure Lie algebra of any proper foliation with TRG is zero, and $\mathcal{A}_B(M,F)$ is a Lie group (Corollary 1).

Examples of computations of the basic automorphism group of a foliation with TRG were constructed. Examples 1 and 2 also show that the group $\mathcal{A}_B(M,F)$ depends on the transverse rigid geometry of the foliation (M,F).

II. Rigid geometries

Parallelizable manifolds Recall that a manifold admitted an e-structure is called *parallelizable.* In other words, a parallelizable manifold is a pair (P,ω), where P is a smooth manifold and ω is a smooth non-degenerate $\mathbb{R}^m$-valued 1-form ω on P, i. e., $\omega_u\colon T_uP \to R^m$ is an isomorphism of the vector spaces for each $u \in P$. Here $m = \dim P$.

Rigid structures Denote by $P(N,H)$ a principal H-bundle with the projection $p\colon P \to N$. Suppose that the action of H on P is a right action and R_a is the diffeomorphism of P, corresponding to an element $a \in H$.

Two principal bundles $P(N,H)$ and $\tilde{P}(\tilde{N},\tilde{H})$ are called *isomorphic* if $H = \tilde{H}$ and there exists a diffeomorphism $\Gamma\colon P \to \tilde{P}$ such that $\Gamma \circ R_a = R_a \circ \Gamma$, $\forall a \in H$.

Definition 1. Let $P(N,H)$ be a principal H-bundle and (P,ω) be a parallelizable manifold satisfying the following condition:

(S) there is an inclusion $\mathfrak{h} \subset \mathbb{R}^m$ of the vector space of the Lie algebra $\mathfrak{h}$ of the Lie group H into vector space $\mathbb{R}^m$ such that $\omega(A^*) = A$, $\forall A \in \mathfrak{h}$, where A^* is the fundamental vector field on P corresponding to A.

Then $\xi = (P(N,H),\omega)$ is called a *rigid structure* on the manifold N. A pair (N,ξ) is called a *rigid geometry.*

Definition 2. Let $\xi = (P(N, H), \omega)$ and $\tilde{\xi} = (\tilde{P}(\tilde{N}, \tilde{H}), \tilde{\omega})$ be two rigid structures. An isomorphism $\Gamma\colon P \to \tilde{P}$ of the principal bundles $P(N, H)$ and $\tilde{P}(\tilde{N}, \tilde{H})$ satisfying the equality $\Gamma^*\tilde{\omega} = \omega$ is called an *isomorphism* of the rigid structures ξ and $\tilde{\xi}$. Any isomorphism Γ of rigid structures ξ and $\tilde{\xi}$ defines a map $\gamma\colon N \to \tilde{N}$ such that $p \circ \Gamma = \gamma \circ p$, and γ is a diffeomorphism from N to $\tilde{N}$. The projection γ is called an *isomorphism* of the rigid geometries (N, ξ) and $(\tilde{N}, \tilde{\xi})$.

Induced rigid geometries Let $\xi = (P(N, H), \omega)$ be a rigid structure on a manifold N with the projection $p\colon P \to N$. Let V be an arbitrary open subset of the manifold N, let $P_V := p^{-1}(V)$ and $\omega_V := \omega|_{P_V}$. Then $\xi_V := (P_V(V, H), \omega_V)$ is also a rigid structure.

Definition 3. The pair (V, ξ_V) defined above is called an *induced rigid geometry on the open subset* V of N.

Effectiveness of rigid geometries Let $\mathcal{A}(\xi)$ be the group of all automorphisms of a rigid structure $\xi = (P(N, H), \omega)$. It is a Lie group as a closed subgroup of the group $\mathcal{A}(P, \omega)$ of all automorphism of a parallelizable manifold (P, ω). Denote by $\mathcal{A}(N, \xi)$ the group of all automorphisms of the geometry (N, ξ), i. e., $\mathcal{A}(N, \xi) := \{\gamma \in \mathrm{Diff}(N) \mid \exists\, \Gamma \in \mathcal{A}(\xi) : p \circ \Gamma = \gamma \circ p\}$. Consider the natural group epimorphism $\chi\colon \mathcal{A}(\xi) \to \mathcal{A}(N, \xi)\colon \Gamma \mapsto \gamma$, where γ is the projection of Γ with respect to $p\colon P \to N$.

Definition 4. Let $\xi = (P(N, H), \omega)$ be a rigid structure on a manifold N with the projection $p\colon P \to N$. The group $\mathrm{Gauge}(\xi) := \{\Gamma \in \mathcal{A}(\xi) \mid p \circ \Gamma = p\}$ is called a *group of gauge transformations of the rigid structure* ξ.

Remark that $\mathrm{Gauge}(\xi)$ is a closed normal Lie subgroup of the Lie group $\mathcal{A}(\xi)$, because it is the kernel of the group epimorphism $\chi\colon \mathcal{A}(\xi) \to \mathcal{A}(N, \xi)$.

Definition 5. A rigid structure $\xi = (P(N, H), \omega)$ is called *effective* if for an arbitrary open subset V in N the induced rigid structure $\xi_V = (P_V(V, H), \omega_V)$ has the trivial group of gauge transformations, i. e., $\mathrm{Gauge}(\xi_V) = \{\mathrm{id}_{P_V}\}$. A rigid geometry (N, ξ) is said *to be effective* if ξ is an effective structure.

Pseudogroup of local automorphisms Let (N, ξ) be a rigid geometry. For arbitrary open subsets V, $V' \subset N$ an isomorphism $V \to V'$ of the induced rigid geometries (V, ξ_V) and $(V', \xi_{V'})$ is called a *local automorphism* of (N, ξ). The family $\mathcal{H}$ of all local automorphisms of a rigid geometry (N, ξ) forms a pseudogroup of local automorphisms. Denote it by $\mathcal{H} = \mathcal{H}(N, \xi)$. Recall that a pseudogroup $\mathcal{H}$ of local diffeomorphisms of manifold N is called *quasi-analytic* if the existence of an open subset $V \subset N$ and an element $\gamma \in \mathcal{H}$ such that $\gamma|_V = \mathrm{id}_V$ implies that $\gamma|_{D(\gamma)} = \mathrm{id}_{D(\gamma)}$ in the entire (connected) domain $D(\gamma)$ on which γ is defined.

Proposition 1. *The pseudogroup $\mathcal{H} = \mathcal{H}(N, \xi)$ of all local automorphisms of an effective rigid geometry (N, ξ) is quasi-analytic.*

III. Foliations with transverse rigid geometries. Foliated bundles

Foliations with transverse rigid geometries (TRG) A foliation (M, F) of codimension q on an n-manifold M has a transverse rigid geometry (N, ξ), where N is a q-manifold, if (M, F) is defined by a cocycle $\eta = \{U_i, f_i, \{\gamma_{ij}\}\}$ modeled on (N, ξ), i. e.,

1) $\{U_i\}$ is an open covering of M;

2) $f_i \colon U_i \to N$ are submersions with connected fibres;

3) $\gamma_{ij} \circ f_j = f_i$ on $U_i \cap U_j$,

with γ_{ij} is a local automorphism of (N, ξ). The topological space N is not assumed to be connected.

Without loss of generality, we will suppose that $N = \cup_{i \in J} f_i(U_i)$ and the family $\{(U_i, f_i)\}$ is maximal as it is generally used in manifold theory.

Definition 6. The rigid geometry (N, ξ) mentioned above is called a *transverse geometry of the foliation* (M, F). The cocycle η modelled on (N, ξ) is said to be an (N, ξ)*-cocycle.*

Assumptions *In this work we will assume that each rigid geometry is effective and all the foliations under consideration are modeled on effective rigid geometries.*

Notations We denote by $\mathfrak{X}(N)$ the Lie algebra of smooth vector fields on a manifold N. If Q is a smooth distribution on M, then $\mathfrak{X}_Q(M) := \{X \in$

$\mathfrak{X}(M) \mid X_u \in Q_u, \forall u \in M\}$. If Q is an integrable distribution and defines a foliation F, where $Q = TF$, we also use notation $\mathfrak{X}_F(M)$ for $\mathfrak{X}_Q(M)$.

Foliated bundles We constructed a foliated bundle for a foliation with TRG and studied its properties.

Theorem 1. *Let (M, F) be a foliation with a transverse rigid geometry (N, ξ), where $\xi = (P(N, H), \omega)$. Then there exist a principal H-bundle $\pi\colon \mathcal{R} \to M$, an H-invariant foliation $(\mathcal{R}, \mathcal{F})$ whose leaves are projected by π onto the leaves of (M, F) and an $\mathbb{R}^m$-valued 1-form $\tilde{\omega}$ on $\mathcal{R}$, where $m = \dim P$, that satisfy the following conditions:*

(i) the map $\tilde{\omega}_u\colon T_u(\mathcal{R}) \to \mathbb{R}^m$, $\forall u \in \mathcal{R}$, is surjective; moreover, $\ker \tilde{\omega}_u = T_u\mathcal{F}$;

(ii) there is an inclusion $\mathfrak{h} \subset \mathbb{R}^m$ of the vector space of the Lie algebra $\mathfrak{h}$ of the Lie group H into $\mathbb{R}^m$ such that $\tilde{\omega}(A^*) = A$, $\forall A \in \mathfrak{h}$, where A^* is the fundamental vector field on $\mathcal{R}$ corresponding to A;

(iii) the foliation $(\mathcal{R}, \mathcal{F})$ is an e-foliation;

(iv) the restriction $\pi_{\mathcal{L}}$ on an arbitrary leaf $\mathcal{L}$ of the foliation $(\mathcal{R}, \mathcal{F})$ is a regular covering map onto a leaf of (M, F), and the subgroup $H(\mathcal{L}) := \{a \in H \mid R_a(\mathcal{L}) = \mathcal{L}\}$ of the Lie group H is the group of deck transformations.

Definition 7. A principal H-bundle $\mathcal{R}(M, H)$ with an H-invariant foliation $(\mathcal{R}, \mathcal{F})$ satisfying the statement of Theorem 1 is called a *foliated bundle for the foliation (M, F) with transverse rigid geometry* (N, ξ) and $(\mathcal{R}, \mathcal{F})$ is called a *lifted foliation.*

If H is disconnected, $\mathcal{R}$ may be also disconnected. In this case all the connected components of $\mathcal{R}$ are mutually diffeomorphic, and we will consider one of them. Thus, we assume that the space of the foliated bundle $\mathcal{R}$ is connected.

IV. Completeness and a structure Lie algebra of a foliation with TRG

Completeness of foliations with TRG Let (M, F) be an arbitrary smooth foliation on a manifold M and TF be the distribution on M formed by the vector spaces tangent to the leaves of the foliation F. The vector quotient bundle TM/TF is called the transverse vector bundle of the foliation (M, F). Let us identify TM/TF with an arbitrary smooth distribution $\mathfrak{M}$ on M that is transverse to the foliation (M, F), i. e., $TM = TF \oplus \mathfrak{M}$.

Let (M, F) be a foliation with TRG and $(\mathcal{R}, \mathcal{F})$ be the lifted foliation. It is natural to identify the transverse vector bundle $T\mathcal{R}/T\mathcal{F}$ with a distribution $\overline{\mathfrak{M}} := \pi^*\mathfrak{M}$ on $\mathcal{R}$, i. e., with a distribution defined by the equality $\overline{\mathfrak{M}}_u := \{X_u \in T_u\mathcal{R} \mid \pi_* X_u \in \mathfrak{M}_x\}$, where $x = \pi(u)$ and $u \in \mathcal{R}$.

Definition 8. A foliation (M, F) with transverse rigid geometry is said to be $\mathfrak{M}$-*complete* if any vector field $X \in \mathfrak{X}_{\overline{\mathfrak{M}}}(\mathcal{R})$ such that $\tilde{\omega}(X) = \mathrm{const}$ is complete. A foliation (M, F) with TRG of arbitrary codimension q is said to be *complete* if there exists a smooth q-dimensional transverse distribution $\mathfrak{M}$ on M such that (M, F) is $\mathfrak{M}$-complete.

In other words, (M, F) is an $\mathfrak{M}$-complete foliation iff the lifted e-foliation $(\mathcal{R}, \mathcal{F})$ is complete with respect to the distribution $\overline{\mathfrak{M}}$ in the sense of Conlon [10]. Remark that complete e-foliation in the sense of Conlon is also complete in the sense of Molino [9].

Proposition 2. *If (M, F) is an $\mathfrak{M}$-complete foliation with TRG, then $\mathfrak{M}$ is an Ehresmann connection for this foliation in sense of Blumenthal and Hebda* [11].

It is well known [10, 9] that for a complete e-foliation $(\mathcal{R}, \mathcal{F})$ all leaves are mutually diffeomorphic.

Structure Lie algebra We applied the relevant results of Molino [9] on complete e-foliations and obtained the following theorem.

Theorem 2. *Let (M, F) be a complete foliation with TRG and $(\mathcal{R}, \mathcal{F})$ be its lifted e-foliation. Then:*

(i) the closure of the leaves of the foliation $\mathcal{F}$ are fibers of a certain locally trivial fibration $\pi_b \colon \mathcal{R} \to W$;

(ii) the foliation $(\overline{\mathcal{L}}, \mathcal{F}|_{\overline{\mathcal{L}}})$ induced on the closure $\overline{\mathcal{L}}$ is a Lie foliation with dense leaves with the structure Lie algebra $\mathfrak{g}_0$, that is the same for any $\mathcal{L} \in \mathcal{F}$.

Definition 9. The structure Lie algebra $\mathfrak{g}_0$ of the Lie foliation $(\overline{\mathcal{L}}, \mathcal{F}|_{\overline{\mathcal{L}}})$ is called a *structure Lie algebra* of the complete foliation (M, F) and is denoted by $\mathfrak{g}_0 = \mathfrak{g}_0(M, F)$.

If (M, F) is a Riemannian foliation on a compact manifold, this notion coincides with the notion of a structure Lie algebra in sense of Molino [9].

V. Category of foliations with TRG

Category of foliations Denote by $\mathfrak{Fol}$ the category of foliations, objects of which are foliations, morphisms of two arbitrary foliations (M, F) and (M', F') are smooth maps $M \to M'$ mapping leaves of the foliation (M, F) into leaves of the foliation (M', F'); a composition of morphisms coincides with the composition of maps.

Category of foliations with TRG Let (M, F) and (M', F') are foliations with transverse rigid geometries (N, ξ) and (N', ξ') defined by an (N, ξ)-cocycle $\eta = \{U_i, f_i, \{\gamma_{ij}\}\}$ and an (N', ξ')-cocycle $\eta' = \{U'_r, f'_r, \{\gamma'_{rs}\}\}$, respectively. Let $f\colon M \to M'$ be a morphism which is a local isomorphism in the category $\mathfrak{Fol}$. Hence for any $x \in M$ and $y := f(x)$ there exist neighborhoods $U_k \ni x$ and $U'_k \ni y$ from η and η' respectively and a diffeomorphism $\lambda\colon V_k \to V'_s$, where $V_k := f_k(U_k)$ and $V'_s := f'_s(U'_s)$, satisfying the relations $f(U_k) = U'_s$ and $\lambda \circ f_k = f'_s \circ f|_{U_k}$. We will say that f preserves *transverse rigid structure* if the diffeomorphism $\lambda\colon V_k \to V'_s$ is an isomorphism of the induced rigid geometries (V_k, ξ_{V_k}) and $(V'_s, \xi'_{V'_s})$.

This notion is well defined, i. e., it does not depend of the choice of neighborhoods U_k and U'_k from the cocycles η and η'.

By a *TRG-morphism of two foliations* (M, F) *and* (M', F') *with transverse rigid geometries* we mean a morphism $f\colon M \to M'$ in the category $\mathfrak{Fol}$ which preserves transverse rigid structure. The category $\mathfrak{F}_{TRG}$ objects of which are foliations with TRG, morphisms are TRG-morphisms, is called *the category of foliations with transverse rigid geometries.*

The following statement shows that the structure Lie algebra $\mathfrak{g}_0(M, F)$ of a foliation (M, F) with TRG is an invariant in the category $\mathfrak{F}_{TRG}$.

Proposition 3. *Let* (M, F) *and* (M', F') *be two foliations with TRG iso-*

morphic in the category $\mathfrak{F}_{TRG}$. Then their structure Lie algebras $\mathfrak{g}_0(M,F)$ and $\mathfrak{g}_0(M',F')$ are isomorphic.

Automorphism groups of foliations with TRG Let (M,F) be a foliation with a fixed transverse rigid structure (N,ξ). Denote by $\mathcal{A}(M,F)$ the group of all automorphisms of (M,F) in the category $\mathfrak{F}_{TRG}$. We say also that $\mathcal{A}(M,F)$ is the *full group of automorphisms.*

Theorem 3. *Let (M,F) be a foliation with TRG. Let $(\mathcal{R},\mathcal{F})$ be the lifted foliation and $\mathcal{A}^H(\mathcal{R},\mathcal{F}) = \{f \in \mathcal{A}(\mathcal{R},\mathcal{F}) \mid f \circ R_a = R_a \circ f, \quad \forall a \in H\}$. Then the map $\mu\colon \mathcal{A}^H(\mathcal{R},\mathcal{F}) \to \mathcal{A}(M,F)\colon \hat{f} \mapsto f$, where f is the projection of $\hat{f} \in \mathcal{A}^H(\mathcal{R},\mathcal{F})$ with respect to $\pi\colon \mathcal{R} \to M$, is a natural group isomorphism.*

VI. Different interpretations of holonomy groups

Equivalent approaches to the notion of holonomy groups Denote by $\Gamma(L,x)$ the germ holonomy group of a leaf L of a smooth foliation (M,F) which is generally used in foliation theory.

Blumenthal and Hebda [11] introduced a notion of a holonomy group of the leaf L of the foliation (M,F) with the Ehresmann connection $\mathfrak{M}$. This group is called an $\mathfrak{M}$-holonomy group and is denoted by $H_{\mathfrak{M}}(L,x)$, $x \in L$ [12]. The following assertion is a direct consequence of Theorem 7 proved by the author in [12].

Theorem 4. *Let (M,F) be a foliation with an Ehresmann connection $\mathfrak{M}$. The natural group epimorphism $\delta\colon H_{\mathfrak{M}}(L,x) \to \Gamma(L,x)$ is an isomorphism if and only if the holonomy pseudogroup of the foliation (M,F) is quasi-analytic.*

We applied Theorems 1 and 4 and proved the following statement about different interpretations of holonomy groups of complete foliations with transverse rigid geometries.

Theorem 5. *Let (M,F) be an $\mathfrak{M}$-complete foliation with TRG defined by an (N,ξ)-cocycle $\{U_i, f_i, \{\gamma_{ij}\}\}$. Let $L = L(x)$, $x \in M$, be an arbitrary leaf of this foliation and $\mathcal{L} = \mathcal{L}(u)$, $u \in \pi^{-1}(x)$, be the corresponding leaf of the lifted foliation $(\mathcal{R},\mathcal{F})$. Then the germ holonomy group $\Gamma(L,x)$ of the leaf L is isomorphic to each of the following five groups:*

(i) the $\mathfrak{M}$-holonomy group $H_{\mathfrak{M}}(L,x)$;

(ii) the group $\mathcal{H}_v$ formed by germs of local diffeomorphisms belonging to the isotropy subpseudogroup of the holonomy pseudogroup $\mathcal{H}$ of local automorphisms of the transverse rigid geometry (N,ξ) at point $v=f_i(x)$, where $x\in U_i$;

(iii) the group of deck transformations of the regular covering map $\pi|_{\mathcal{L}}\colon \mathcal{L}\to L$;

(iv) the subgroup $H(\mathcal{L})=\{a\in H \mid R_a(\mathcal{L})=\mathcal{L}\}$ of the Lie group H;

(v) the holonomy group $\Phi(u)$ of the integrable connection $T(\mathcal{F}|_{\pi^{-1}(L)})$ in the principal H-bundle with the projection $\pi|_{\pi^{-1}(L)}\colon \pi^{-1}(L)\to L$.

VII. The groups of basic automorphisms of foliations with TRG

Let $\mathcal{A}(M,F)$ be the full automorphism group of a foliation (M,F) with TRG. We denote by $\mu\colon \mathcal{A}^H(\mathcal{R},\mathcal{F})\to\mathcal{A}(M,F)$ the group isomorphism defined in Theorem 3.

Definition 10. The quotient group $\mathcal{A}_B(M,F):=\mathcal{A}(M,F)/\mathcal{A}_L(M,F)$ is called the *basic automorphism group* of the foliation (M,F) with TRG.

Emphasize that the basic automorphism group $\mathcal{A}_B(M,F)$ of a foliation (M,F) with TRG is an invariant of this foliation in the category $\mathfrak{F}_{TRG}$.

Theorem 6. *Let (M,F) be a foliation with TRG and $(\mathcal{R},\mathcal{F})$ be the lifted foliation. Denote by $\mathcal{A}^H_B(\mathcal{R},\mathcal{F})$ the quotient group $\mathcal{A}^H(\mathcal{R},\mathcal{F})/\mathcal{A}^H_L(\mathcal{R},\mathcal{F})$. There exists a natural group isomorphism $\chi\colon\mathcal{A}^H_B(\mathcal{R},\mathcal{F})\to\mathcal{A}_B(M,F)$ satisfying the equality $s\circ\mu=\chi\circ r$, where $r\colon\mathcal{A}^H(\mathcal{R},\mathcal{F})\to\mathcal{A}^H_B(\mathcal{R},\mathcal{F})$ and $s\colon\mathcal{A}(M,F)\to\mathcal{A}_B(M,F)$ are the associated group epimorphisms onto the quotient groups.*

IX. Conditions guarantee that $\mathcal{A}_B(M,F)$ is a Lie group

The case $\mathfrak{g}_0(M,F)=0$ A leaf L of a foliation (M,F) is called *closed* if L is a closed subset in the topology of the manifold M. Further we use the term “a closed leaf” only in this sense.

Theorem 7. *Let (M, F) be a complete foliation with a transverse rigid geometry (N, ξ), where $\xi = (P(N, H), \omega)$. Suppose that the structure Lie algebra $\mathfrak{g}_0(M, F)$ is zero. Then:*

(i) the full basic automorphism group $\mathcal{A}_B(M, F)$ admits a Lie group structure with the following estimate of its dimension:

$$\dim \mathcal{A}_B(M, F) \leq \dim P; \tag{1}$$

(ii) if there exists an isolated closed leaf L of the foliation (M, F), then

$$\dim \mathcal{A}_B(M, F) \leq \dim H; \tag{2}$$

(iii) there exists a unique topology and a unique smooth structure on the full group $\mathcal{A}_B(M, F)$ of basic automorphisms of the foliation (M, F), making $\mathcal{A}_B(M, F)$ into a Lie group.

Theorem 7 does not exclude the triviality of the full group $\mathcal{A}_B(M, F)$.

Remark 1. The main result of the work [3] by Belko is the theorem asserting that if there exists a closed leaf of a foliation (M, F) with complete transversally projectable affine connection, then the group $\mathcal{A}_B(M, F)$ is a Lie group. This statement is not correct. It's proof essentially uses the fact that existence of a closed leaf of this foliation implies that the lifted foliation is simple. It is not true, in general. Let us consider a foliation (M, F) from Example 3 (in Section X), when $r = 1/\pi$, as affine foliation. It has a compact leaf, but $\mathfrak{g}_0(M, F) = \mathbb{R}^1 \not\equiv 0$, hence the lifted foliation is not simple. Thus the foliation (M, F) is a Lie foliation with non-zero structure Lie algebra $\mathfrak{g}_0(M, F)$. Hence the group $\mathcal{A}_B(M, F)$ is not a Lie group.

Discrete holonomy groups of leaves Let (M, F) be a foliation with TRG. Let $\pi \colon \mathcal{R} \to M$ be the projection of the foliated bundle over (M, F).

Definition 11. We say that the holonomy group of a leaf $L \ni x$ of the foliation (M, F) is *discrete* if there exists a point $u \in \pi^{-1}(x)$ such that the group $H(\mathcal{L}) := \{a \in H \mid R_a(\mathcal{L}) = \mathcal{L},\ \mathcal{L} = \mathcal{L}(u) \in \mathcal{F}\}$ is a discrete subgroup of the Lie group H.

Let $u' \in \pi^{-1}(x)$ and $u \notin \mathcal{L}' = \mathcal{L}'(u')$. In this case the subgroup $H(\mathcal{L}')$ is conjugate to the subgroup $H(\mathcal{L})$ in the Lie group H. Hence $H(\mathcal{L})$ is a discrete subgroup of H if and only if $H(\mathcal{L}')$ is a discrete subgroup of H. Thus, according to Theorem 5 the notion of discrete holonomy group of leaf L is well defined.

Recall that a leaf L of a foliation (M, F) is said to be *proper* if L is an embedded submanifold in M. A foliation (M, F) is called *proper* if each its leaf is proper.

Let (M, F) be a complete foliation with TRG. We proved that the existence a proper leaf L with a discrete holonomy group implies that the structure Lie algebra $\mathfrak{g}_0(M, F)$ is zero. In view of this fact and Theorem 7 we got the following statement.

Theorem 8. *Let (M, F) be a complete foliation with transverse rigid geometry (N, ξ), where $\xi = (P(N, H), \omega)$. If at least one of the following conditions holds:*

(i) there exists a proper leaf L with discrete holonomy group;

(ii) there is a closed leaf L with discrete holonomy group;

(iii) there exists a proper leaf L with finite holonomy group;

(iv) there is a closed leaf L with finite holonomy group,

then the basic automorphism group $\mathcal{A}_B(M, F)$ admits a Lie group structure of dimension at most $\dim P$, *and this structure is unique.*

It is well known that any foliation has leaves without holonomy. Hence the following statement is a consequence of Theorem 8.

Corollary 1. *For any proper complete foliation (M, F) with TRG the basic automorphism group $\mathcal{A}_B(M, F)$ admits a unique Lie group structure.*

X. Examples

Suspended foliations The suspension of a homomorphism was suggested by Haefliger. This method of construction examples is widely used in foliation theory.

Let $\rho\colon \pi_1(B, b_0) \to \mathrm{Diff}(T)$ be a homomorphism of the fundamental group of a manifold $B \ni b_0$ into the group of diffeomorphisms of a q-dimensional manifold T, and let $p\colon \hat{B} \to B$ be the universal covering mapping. Then we have a right action of the group $\Pi := \pi_1(B, b_0)$ on $\hat{B}$ by deck transformations. The equality

$$(x,t) \cdot g := (x \cdot g, \rho(g^{-1})(t)), \quad \forall (x,t) \in \hat{B} \times T, \quad \forall g \in \Pi,$$

defines a free right properly discontinuous smooth action of the group Π on the product of manifolds $\hat{B} \times T$; therefore the quotient manifold $M := \hat{B} \times_{\Pi} T$ is defined. Let $\kappa\colon \hat{B} \times T \to M$ be the natural projection. Then $F := \{\kappa(\hat{B} \times \{t\}) \mid t \in T\}$ is a foliation of codimension q on M; in this case, it is said that the *foliation* (M,F) *is obtained by suspension of the homomorphism* ρ. For this foliation we will use the notation $(M,F) := \mathrm{Sus}(T, B, \rho)$. The image $\Psi := \mathrm{im}\rho$ is the global holonomy group of (M,F).

Transversally similar and transversally homothetic foliations Let G be the similarity group of the Euclidean space $\mathbb{E}^q$, $q \geq 1$, and $\mathbb{R}^+$ be the multiplicative group of positive real numbers. Then $G = CO(q) \leftthreetimes \mathbb{R}^q$ is the semidirect product of the conformal group $CO(q) = \mathbb{R}^+ \cdot O(q)$ and the group $\mathbb{R}^q$. Let $H = CO(q)$ and $p\colon G \to G/H = \mathbb{E}^q$ be the canonical principal H-bundle. Let $\mathfrak{g}$ be the Lie algebra of the Lie group G, and ω be the Maurer-Cartan $\mathfrak{g}$-valued 1-form on G. Then $\xi = (G(\mathbb{E}^q, H), \omega)$ is an effective rigid geometry. Foliations with this transverse geometry $(\mathbb{E}^q, \xi)$ are called *transversally similarity foliations* [7].

Denote by E the neutral element of the group $O(q)$. If $G = (\mathbb{R}^+ \cdot E) \leftthreetimes \mathbb{R}^q$, $H = \mathbb{R}^+ \cdot E$, and ω is the Maurer-Cartan $\mathfrak{g}$-valued 1-form on the Lie group G, then foliations with the transverse effective rigid geometry $(\mathbb{E}^q, \xi)$, where $\xi = (G(\mathbb{E}^q, \mathbb{R}^+ \cdot E), \omega)$, are called *transversally homothetic foliations* [7].

Example 1. Let B be a smooth p-dimensional manifold whose fundamental group $\pi_1(B, b)$ contains an element α of infinite order. For an arbitrary natural number $q \geq 1$, denote by $\mathbb{E}^q$ a q-dimensional Euclidean space. Define a homomorphism $\rho\colon \Pi := \pi_1(B, b) \to \mathrm{Diff}(\mathbb{E}^q)$ by setting $\rho(\alpha) = \psi$, where ψ is the homothetic transformation of the Euclidean space $\mathbb{E}^q$ with the coefficient $\lambda \not\equiv 1$, i. e. $\psi(x) = \lambda x$, $\forall x \in \mathbb{E}^q$, and $\rho(\beta) = \mathrm{id}_{\mathbb{E}^q}$ for any element $\beta \in \pi_1(B, b)$ such that $\beta \not\equiv \alpha^k$ with some integer k. Then $(M,F) = \mathrm{Sus}(E^q, B, \rho)$ is a proper transversally similar foliation with a unique closed leaf diffeomorphic to B.

According to Corollary 1, the full basic automorphism group $\mathcal{A}_B(M,F)$ of this foliation (M,F) admits a unique Lie group structure. The group $\mathcal{A}(\xi)$ is equal to the group of left translations of the Lie group $G = CO(q) \leftthreetimes \mathbb{R}^q$, hence we can identify $\mathcal{A}(\mathbb{E}^q,\xi) \cong \mathcal{A}(\xi)$ with G. In this case it is not difficult to show that the full group of basic automorphisms $\mathcal{A}_B(M,F)$ is isomorphic to the quotient group $N(\Psi)/\Psi$, where $N(\Psi)$ is the normalizer of Ψ in the Lie group G. In our case $\Psi = \langle\psi\rangle$ and $N(\Psi) = \mathbb{R}^+ \cdot O(q)$, therefore $\mathcal{A}_B(M,F) \cong U(1) \times O(q)$, where $U(1) \cong (\mathbb{R}^+ \cdot E)/\Psi$ is the compact 1-dimensional abelian group.

If $q = 1$, then $O(q) = \mathbb{Z}_2$ and $\mathcal{A}_B(M,F) \cong U(1) \times \mathbb{Z}_2$.

Example 2. Consider the foliation (M,F) constructed in Example 1 as a transversally homothetic foliation, i. e., with a different transverse rigid geometry. In this case the Lie group $\mathcal{A}_B(M,F)$ is isomorphic to the quotient Lie group $N(\Psi)/\Psi$, where $N(\Psi)$ is the normalizer of Ψ in the Lie group $(\mathbb{R}^+ \cdot E) \leftthreetimes \mathbb{R}^q$. Since $N(\Psi) = \mathbb{R}^+ \cdot E$, so $\mathcal{A}_B(M,F) \cong U(1)$.

Remark 2. In both examples 1 and 2 the foliation (M,F) has a unique closed leaf and, in Theorem 7, the equality is achieved in the estimate (ii) of the dimension of $\mathcal{A}_B(M,F)$.

Example 3. Let ψ be the rotation of the plane $\mathbb{E}^2$ about the point $0 \in \mathbb{E}^2$ through the angle $\delta = 2\pi r$. Consider an Euclidean metric g on $\mathbb{E}^2$. Denote by $Iso(\mathbb{E}^2,g)$ the full isometry group of $(\mathbb{E}^2,g)$. Let $\rho\colon \pi_1(S^1,b) \cong \mathbb{Z} \to Iso(\mathbb{E}^2,g)$ be defined by the equality $\rho(1) := \psi$, $1 \in \mathbb{Z}$. Then we have a suspended Riemannian foliation $(M,F) := \mathrm{Sus}(\mathbb{E}^2,S^1,\rho)$. This foliation has a unique closed (compact) leaf.

There exists a group isomorphism between $\mathcal{A}_B(M,F)$ and the quotient group $N(\Psi)/\Psi$, where $\Psi = \langle\psi\rangle$ and $N(\Psi)$ is the normalizer of Ψ in the Lie group $Iso(\mathbb{E}^2,g)$ identified with $O(2) \leftthreetimes \mathbb{R}^2$. Since $N(\Psi) = O(2)$, so $\mathcal{A}_B(M,F) = O(2)/\Psi$. Hence $\mathcal{A}_B(M,F)$ admits a Lie group structure if and only if Ψ is a closed subgroup of $O(2)$ or, equivalent, when $\delta = 2\pi r$ for some rational number r. If $\delta = 2\pi r$, where r is a nonzero rational number, then $\mathcal{A}_B(M,F) \cong O(2)$.

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CORRELATIONS BEETWEN THE QUANTUM FLUCTUATIONS AND THE PHASE OF THE GRAVITATIONAL WAVES

E. Balogh[1], I. Bartos[2], I. Lovas[3] and Sz. Márka[2]

[1] János Bolyai Gymnasium, Salgótarján, Hungary
[2] Columbia University, Department of Physics, New York, NY10027, USA
[3] Debrecen University, Department of Theoretical Physics, Debrecen, H4010, Hungary

Abstract

The existence of gravitational waves is proved by astronomical observations. The belief that the gravitational waves are quantized is almost hundred years old. Nevertheless up till now there are neither theoretical, nor observational proof of this belief. In this note we suggest to measure the fluctuations of the gravitational waves. If the fluctuations are correlated with the phase of the gravitational wave, in other words, if the gravitational wave is squeezed, then it is quantized.

I. Introduction

The Einstein-equations of the general relativity can be reduced to wave equations in linear approximation. It was taken as granted that the solutions of these equations describe gravitational waves, which exist in nature. Moreover it was assumed already before the middle of the last century that these waves are quantized, i.e. they are associated with gravitons having energy of $h\nu$ and spin of $2h/(2\pi)$, where the Planck-constant is denoted by h and the frequency by ν. The existence of the gravitational waves, however, was proved only in the second part of the XX-th century by Hulse and

Taylor [1]. They observed the pulsar radiation of a neutron star which is moving around another neutron star. It was possible to observe that the periastron is shifted in a similar way as in the case of the Mercury moving around the Sun, furthermore it was seen that the energy of the system is decreasing continuously. Both phenomena were perfectly described by the Einstein-equations if the possibility of gravitational wave emission was taken into account.

Recently a binary system of huge black holes has been observed in the J 287 quasar [2]. A very spectacular outburst is produced by the smaller black hole when it collides with the accretion disc of the bigger black hole. A great number of outbursts were observed and interpreted correctly. If the emission of the gravitational waves were neglected from the analysis the beautiful agreement was destroyed. Thus one may conclude that the motion of the binary system with mass 17 billion Sun mass can be described perfectly well by the Einstein-equations, and the gravitational waves really exist in nature. Since now we are convinced about the existence of the gravitational waves it is justified to hope that they can be observed sooner or later on the surface of the Earth, as well.

The question of the quantized character of the gravitational waves is a more complicated issue. Up till now the quantization of the theory of gravitation is an unresolved problem in spite of the tremendous amount of efforts. Consequently the theoretical proof of the quantized character of the gravitational waves is missing. The experimental proof of the quantized character is missing either.

In this note we try to find a possibility to observe the quantized character of the gravitational waves. We assume that the basic features of the quantization of the gravitation are similar to that of the electromagnetism. Therefore we look for genuine, observable signatures of the quantization in the realm of electromagnetism. The energy quantum $h\nu$ belongs to this category, however it can not be used in the case of the gravitation because of the extremely low values of the frequency ν. It was proved by Glauber [3] in the framework of the quantum electrodynamics that the phenomenon of the squeezing is a genuine signature of the quantized nature. The existence of the squeezing was proved by experiments, that is, definite correlation has been found between the phase of the wave and the quantum fluctuations.

Here we assume that something similar is true in the case of the gravitational waves, as well. It was pointed out by Grishchuk [4] that the quantum noise is correlated with the phase of the gravitational wave if it is generated by the non-linear gravitational background. He focused the attention to those gravitational waves which were generated in the time of the Big Bang. Here we want to emphasize that those existing and working GW detectors which will be able to detect the arrival of the gravitational waves will be able to detect also the quantum fluctuations. If some correlation can be observed between the phase of the wave and the quantum fluctuations then this can be considered as a proof for the quantized character of the gravitational waves [5]. If no correlation can be found then we are not able to draw any kind of conclusion.

II. Analysis of the signal arriving from the interferometer type gravitational wave detector

We assume that the light signal $L(t)$ arriving from the interferometer type gravitational wave detector at time t can be described by the following sum:

$$L(t) = C(t,\nu) + Q(t,\nu,\varphi) + B(t); \tag{1}$$

where the frequency of the wave is denoted by ν, the contribution of the "classical wave" by $C(t,\nu)$, the contribution of the "quantum fluctuation" by $Q(t,\nu,\varphi)$ and the contribution of the external random background by $B(t)$. In the first step of the analysis we neglect $Q(t,\nu,\varphi)$, and we determine from the observed data the quantities ν, $C(t,\nu)$, and $B(t)$. By the way, this is the original task of the gravitational wave detector! As a second step of the analysis we calculate from the observed data the contribution of the quantum fluctuations $Q(t,\nu,\varphi)$Q, using the values of ν, $C(t,\nu)$, and $B(t)$, obtained in the first step of the analysis.

III. The Energy Flux of the Gravitational Waves

We consider a gravitational plane wave far away from its source having frequency ν and amplitude a. The energy flux F of such a wave, i.e. the energy per unit area, per sampling (with sampling frequency ν_s) can be expressed in the following form:

$$F = \frac{\pi c^3 \nu^2 a^2}{G \nu_s}. \tag{2}$$

The energy arriving into the detector per sampling is given by:

$$E = FA, \tag{3}$$

where A is the cross sectional area of the detector.

The expectation value of the number of the gravitons arriving into the detector per sampling is given by:

$$\langle N \rangle = \frac{E}{h\nu} = \frac{\pi A c^3 \nu^2 a^2}{h G \nu_s}. \tag{4}$$

The fluctuation of the graviton number may be approximated by the following expression [6] :

$$\langle \Delta N^2 \rangle = \langle N \rangle \left[e^{-2S} \cos^2 \left(\frac{\varphi}{2} - \theta \right) + e^{2S} \sin^2 \left(\frac{\varphi}{2} - \theta \right) \right]. \tag{5}$$

Here θ is the phase of the wave at time t: $\theta = \nu t + \theta_0$.

The squeezing parameters are denoted by S and φ. The lack of squeezing is characterised by $S = 0$. In this case the number of gravitons is described by the Poisson-distribution:

$$\langle \Delta N^2 \rangle = \langle N \rangle. \tag{6}$$

If the measured values of the signal are stored together with a time stamp by a Field Programmable Gate Array (FPGA) [7], then the evaluation of the measurement can be done off line. The evaluation can be performed as an iterative procedure when the stored values can be used repeatedly. It is worth while to point out that by using an FPGA the comparison of the signals of parallel detectors can be done also off line. If the noise/signal ratio is not too large then the frequency ν, the squeezing parameters S and φ, and the value of the random noise can be obtained from the measurements. If the value of the squeezing parameter S turns out to be significantly different from zero then, it is proved that *the gravitational waves are quantized!* The success of such an experiment depends first of all on the distance of the source of the gravitational waves. It must be confessed that if the signal contains more then one frequency with non-negligible amplitudes then the analysis will be rather tedious.

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SOME GEOMETRICAL ANALOGUES BETWEEN THE DESCRIPTION OF THE STATES SPACE IN NON-CLASSICAL PHYSICS AND THE EVENTS SPACE IN CLASSICAL PHYSICS ††

O. N. Golubjeva[1], **A. D. Sukhanov**[2]

[1] Russian Peoples Friendship University, Moscow, Russia
[2] Joint Institute for Nuclear Research, Dubna, Russia

Abstract

A certain resemblance between properties of the states space in non-classical Physics and the events space in classical Physics is recognized.

It is noted that in the absence of thermal influence or, correspondingly, of gravitation there are the simplest Riemannian structures with a diagonal metric and zero curvature in both cases. Either the square of the half of the Planck's constant or the square of the electrodynamics constant are the invariants, limiting the minimal values of corresponding quantities. These minimal limitations are initially intrinsic to the object environment only in the form of "cold" vacuum. It is proposed the concepts of "self-action" and "equilibrium shell".

In view of gravitation in the object environment or the change of "cold" vacuum to "warm" one lead to cardinal new properties. First of all, the non-trivial Riemannian structures appear so that the metrics becomes non-diagonal. Second, in both cases the curvature of space becomes not equal to zero.

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One may consider all these circumstances as first steps to join both spaces of matter existence.

CONTENTS

I. Introduction

The main aim of our investigation is to exhaust the maximal physical information from the analysis of geometrical properties of the non-classical states space. Under the term non-classical we understand all situations when an object is under a stochastic influence - both quantum ("cold" vacuum in Quantum Mechanics - QM) and thermal one (thermostat in Statistical Thermodynamics – ST).

In the most general case an object is affected by both types of the influence simultaneously. As a result the characteristics of the object fluctuates. They are said to be quantum-thermal fluctuations. Our study is based on the Cauchy-Bunjakovsky-Schwarz unequality (hereafter noted CBSU) used in the states space.

II. Fluctuation states submanifold of coordinate and momentum

From the Hilbert manifold of arbitrary states $| >$ for a micro-object let us select a submanifold of fluctuations states. For this goal let us introduce first of all the operators of coordinate and momentum fluctuations $\Delta\widehat{q}$ and $\Delta\widehat{p}$ respectively in such a manner:

$$\Delta\widehat{q} = \widehat{q} - \langle|\widehat{q}\,|\rangle; \tag{1}$$

$$\Delta\widehat{p} = \widehat{p} - \langle|\widehat{p}\,|\rangle. \tag{2}$$

One can get the submanifold, that is of interest to us, as a result of the operators $\Delta\widehat{q}$ and $\Delta\widehat{p}$ acting upon an arbitrary state $|>$:

$$|\Delta q\rangle \equiv \Delta\widehat{q}\,|\;\rangle; \tag{3}$$

$$|\Delta p\rangle \equiv \Delta\widehat{p}\,|\;\rangle. \tag{4}$$

We call it the submanifold of fluctuation states of the coordinate and the momentum. As it is well known, in an arbitrary Hilbert space a bilinear hermitian form is defined. Usually it is treated as a scalar product of state vectors $< \Phi|\Psi >$.

In the selected submanifold of fluctuations states it is:

$$R_{pq} \equiv \langle\Delta p|\Delta q\rangle = \langle|\Delta\widehat{p}\,\Delta\widehat{q}\,|\rangle \tag{5}$$

or equivalently

$$R_{p\,q} \equiv \frac{1}{2}\langle|\Delta\widehat{p}\,\Delta\widehat{q}\, + \Delta\widehat{q}\,\Delta\widehat{p}\,|\rangle + \frac{1}{2}\langle|\Delta\widehat{p}\,\Delta\widehat{q}\, - \Delta\widehat{q}\,\Delta\widehat{p}\,|\rangle. \tag{6}$$

As at the same time the scalar product is a complex quantity, it is convenient to write this expression in a different way

$$R_{pq} = \sigma_{p\,q} + i\,c_{p\,q}, \tag{7}$$

where its imaginary part

$$c_{p\,q} \equiv \frac{1}{2}|\langle|\{\widehat{p},\widehat{q}\,\}|\rangle| = \frac{\hbar}{2} \tag{8}$$

characterizes a symplectic structure on the selected states submanifold $|\Delta p >$, $|\Delta q >$ in the Hilbert space.

However, the subject of our subsequent interest will be mainly the real term

$$\sigma_{p\,q} \equiv \frac{1}{2}\langle|\{\Delta\widehat{p},\Delta\widehat{q}\,\}|\rangle. \tag{9}$$

In the quasi-classical limit the operators $\Delta\widehat{p}$ and $\Delta\widehat{q}$ can be changed by c-numbers. In this case the quantity σ_{pq} is in close connection with the standard definition of a correlator in the probability theory. This fact allows us to call σ_{pq} a correlator of quantum or (in more general case) quantum-thermal fluctuations of coordinate and momentum, or, a quantum correlator.

If the quantity σ_{pq} includes the two identical operators $\Delta\widehat{q}$ or $\Delta\widehat{p}$ it takes the form either $\sigma_{qq} = \langle\Delta q|\Delta q\,\rangle = \langle|(\Delta\widehat{q}\,)^2|\rangle$ or $\sigma_{pp} = \langle\Delta p|\Delta p\,\rangle = \langle|(\Delta\widehat{p}\,)^2|\rangle$ where σ_{qq} and σ_{pp} are dispersions of coordinate and momentum. All the three quantities σ_{pq}, σ_{pp}, and σ_{qq} together describe the Riemannian structure on the submanifold of states under study.

Now let us make some remark following Caianiello and Noce [1]. In the frame of ST they supposed that one can interpret the correlator of thermal fluctuations of a conjugated macroparameter pair a and b as a "scalar product" of conventional "vectors" δa and δb in the Riemannian space of the thermal fluctuations

$$\sigma^T_{a\,b} = \overline{(\Delta a \Delta b)} \equiv \delta a \cdot \delta b. \tag{10}$$

Then dispersions of random quantities a and b are

$$\sigma^T_{a\,a} = \overline{(\Delta a)^2} \equiv (\delta a)^2, \tag{11}$$

$$\sigma^T_{b\,b} = \overline{(\Delta b)^2} \equiv (\delta b)^2$$

and have a sense of norms of the "vectors". We once more emphasise that all the three quantities $\sigma^T_{a\,b}$, $\sigma^T_{a\,a}$, and $\sigma^T_{b\,b}$ describe the Riemannian space of thermal fluctuations.

This fact allows us to use this idea in our case. For this goal we introduce the two-dimensional Riemannian space and on this ground we assume that the three quantities, i.e. the quantum correlator σ_{pq} and the dispersions σ_{qq},σ_{pp} can be interpreted as a conventional "scalar product" and norms of peculiar vectors δq and δp in this space:

$$\sigma_{p\,q} \equiv (\delta p\, \delta q), \quad \sigma_{p\,p} \equiv (\delta p)^2, \quad \sigma_{q\,q} \equiv (\delta q)^2. \tag{12}$$

Some additional reason for identifying the quantum correlator with a "scalar product" of conventional "vectors" is the similar behavior of the quantum correlator in the high temperature limit and that of the thermal correlator.

In this case the quantum correlator σ_{pq} is in close connection with the thermal correlator

$$\sigma_{p,q} \to \sigma^T_{p\,q} = \overline{(\Delta p \Delta q)}. \tag{13}$$

III. Schroedinger uncertainty relation

Let us consider some peculiarities of the states submanifold $|\Delta p>$, $|\Delta q>$. Our starting point is the CBSU. We note that in many kinds of manifolds it plays a role of some limiter for the corresponding geometrical structures.

Thus for the given submanifold we have the CBSU in the form which physicists call the Schroedinger uncertainty relation (SUR):

$$(\delta p)^2(\delta q)^2 \geqslant |R_{p\,q}|^2 = \sigma^2_{p\,q} + c^2_{p\,q}. \tag{14}$$

Let us remember that R_{pq} is the transition amplitude from the state $|\Delta q\rangle$ to the state $|\Delta p\rangle$. Thus we see that the squared transition amplitude can not be more then "vector" norms product $(\delta p)^2(\delta q)^2$.

Note, that the transition amplitude has two terms. The second of it c_{pq} reflects a type of correlation between momentum and coordinate related to the non-commutativity of the corresponding operators. At the same time the first of it σ_{pq} in the general case corresponds to another correlation type somewhat analogous to correlation one in the classical probability theory.

Below we restrict ourselves to the analysis of states for which SUR transforms into the strict equality

$$(\delta p)^2(\delta q)^2 = \sigma^2_{p\,q} + c^2_{p\,q}. \tag{15}$$

In Physics such SUR is usually said to be saturated.

IV. A model of "quantum oscillator"

In the given model the saturated SUR has importance in the two cases:
- in the basic state (its wave function is real, $c_{pq} = 0$). This state belongs to the family of coherent states (CS);
- in other states with complex wave functions that satisfy the important

condition $c_{pq} \not\equiv 0$. We call such states correlated-coherent states (CCS). Note, that to pass from CS to CCS for the quantum oscillator it is necessary to use (u, v)- Bogoliubov transformations generating the Lie group SU(1.1).

Among many kinds of CCS there are states that are especially interesting for physicists because they are more close to the real Nature. These are thermal CCS (TCCS) that were first introduced by Umezawa [2] in the frame of his thermofield dynamics (TFD). Complex wave functions describing TCCS for quantum oscillator in a thermostat suppose both quantum and thermal stochastic influence of environment simultaneously [3]. These functions must have a temperature-dependent amplitude and phase.

To study the fluctuations of Riemannian space of momentum and coordinate for quantum oscillator in TCCS in more detail we use SUR below in the saturated form (15). Earlier in the paper [3] we obtained a formula for the wave function for the quantum oscillator in the thermostat:

$$\psi(q) = [2\pi\overline{(\Delta q)^2}]^{-1/4} \exp\left\{-\frac{q^2}{4\overline{(\Delta q)^2}}(1 - i\alpha)\right\}, \tag{16}$$

where

$$\alpha = \left[\sinh \frac{\hbar\omega}{2k_B T}\right]^{-1}.$$

From it one can calculate dispersions of momentum and coordinate at any temperature:

$$(\delta p)^2 = \frac{\hbar m\omega}{2} \coth \frac{\hbar\omega}{2k_B T}, \tag{17}$$

$$(\delta q)^2 = \frac{\hbar}{2m\omega} \coth \frac{\hbar\omega}{2k_B T}. \tag{18}$$

We emphasise that these quantities depending on the wave function amplitude are functions of the temperature. From the formula for the wave function we can obtain also the quantum correlator

$$\sigma_{pq} = \frac{\hbar}{2} \left[\sinh \frac{\hbar\omega}{2k_B T}\right]^{-1}. \tag{19}$$

It depends on the wave function phase and, what is the most important, it is a function of the temperature, too.

In the frame of our Riemannian space we can assume that the expression

$$\frac{\sigma_{p\,q}}{\sqrt{\sigma_{pp}}\,\sqrt{\sigma_{qq}}} \tag{20}$$

is a quantity somewhat analogous to the function $\cos\varphi$ for the usual scalar product in the Euclidian space. We note that using all these formulas and recall (12) as

$$(\delta q)^2 \equiv \sigma_{qq}; \quad (\delta p)^2 \equiv \sigma_{pp} \tag{21}$$

one can easily obtain the conventional $\cos\varphi$.

For the "angle" between "vectors" in the Riemannian space this quantity is equal to

$$\left[\cosh\frac{\hbar\omega}{2k_BT}\right]^{-1}. \tag{22}$$

As a result we obtain that "lengths of vectors" rise while the "angle" between them decreases with increasing temperature. So the conventional $\cos\varphi$ changes in the region from 0 to 1 as it is necessary. Thus if no correlation exists between fluctuations ($\sigma_{pq} = 0$) the "vectors" δp and δq are "orthogonal"(at $T = 0$). In the general case (when the correlator $\sigma_{pq} \not\equiv 0$) the "vectors" have an arbitrary mutual orientation. It maximally approximates collinearity when their scalar product gets the maximal value.

V. Some interesting geometrical characteristics

For convenience of further calculations we make a slight change of variables:

$$(\delta p)^2 = m\omega(\delta\widetilde{p})^2; \tag{23}$$

$$(\delta q)^2 = \frac{1}{m\omega}(\delta\widetilde{q})^2. \tag{24}$$

At the same time the scalar product does not change

$$\sigma_{pq} = \widetilde{\sigma}_{pq}. \tag{25}$$

Taking new variables $\delta\widetilde{p}$ and $\delta\widetilde{q}$ as basic vectors in the Riemannian space we get the SUR for the quantum oscillator in TCCS

$$(\delta\widetilde{p})^2(\delta\widetilde{q})^2 \equiv \frac{\hbar^2}{4}\coth^2\frac{\hbar\omega}{2k_BT} = \tag{26}$$
$$= (\sigma_{pq}^2 + \frac{\hbar^2}{4}) \equiv \frac{\hbar^2}{4}\left[\sinh\frac{\hbar\omega}{2k_BT}\right]^{-2} + \frac{\hbar^2}{4}.$$

So we have introduced a fluctuation space of momentum and coordinate with the basic vectors δp and δq dependent on the temperature. For its analysis we have two possibilities, based on the saturated SUR for the quantum oscillator in a thermostat.

The first solution is as follows. Let us rearrange the term σ_{pq}^2 to the left side of SUR. Now we can consider the combination

$$(\delta\widetilde{p})^2(\delta\widetilde{q})^2 - \sigma_{p\,q}^2 \tag{27}$$

as an entire quantity. From geometrical point of view it is a non-degenerate determinant of some two-dimensional metric tensor g_{ik}:

$$Det\ \ g_{ik} = \begin{Vmatrix} (\delta\widetilde{p})^2 & \sigma_{pq} \\ \sigma_{pq} & (\delta\widetilde{q})^2 \end{Vmatrix}. \tag{28}$$

Taking into account the actual values of quantities $(\delta\widetilde{q})^2$, $(\delta\widetilde{p})^2$, and σ_{pq} one can rewrite this determinant in the form

$$Det\ \ g_{ik} = \frac{\hbar^2}{4}\begin{Vmatrix} \coth\frac{\hbar\omega}{2k_BT} & \left[\sinh\frac{\hbar\omega}{2k_BT}\right]^{-1} \\ \left[\sinh\frac{\hbar\omega}{2k_BT}\right]^{-1} & \coth\frac{\hbar\omega}{2k_BT} \end{Vmatrix}. \tag{29}$$

We can see the following. Although all its components are temperature dependent, $Det\ \ g_{ik} = (\hbar/2)^2$ is obviously independent on the quantity T. In the limit $T \to 0$: $g_{11} \to 1$ and $g_{22} \to 1$, $g_{12} \to 0$ and $g_{21} \to 0$. This fact corresponds to the orthogonality condition of the "vectors" δp and δq.

In the limit $(1/T) \to 0$ $Det\ \ g_{ik}$ does not change. So we can claim the determinant is invariant under Bogoliubov (u, v) - transformations. One may expect that a scalar curvature of corresponding space is not equal to zero at $T \not\equiv 0$ but at $T \to 0$ it reduces to zero.

We can also assume that the equality of values $(c_{pq})^2$ and $Det \quad g_{ik}$ is not by chance. We suppose it reflects a peculiar interference between Riemannian and symplectic structures on the submanifold of fluctuation states of coordinate and momentum. At the same time it can serve as an initial criterion of belonging one or another state to the ÑÑS family.

The second possibility of analysis is connected with the interpretation of the right side of SUR itself. It is common practice to consider the term $\delta p \delta q$ as an entire mathematical quantity, named uncertainty product $(\mathcal{UP})$ hereafter noted

$$(\mathcal{UP}) \equiv \delta p \, \delta q. \tag{30}$$

Earlier we supposed a new theory - Quantum Generalization of equilibrium Statistical Thermodynamics (QGST)[4]. In the case of the quantum oscillator in a thermostat we found the physical sense of $\delta p \delta q$. For this goal we introduced a new macro-parameter - the effective action as an adiabatic invariant

$$J = \frac{\mathcal{E}}{\omega}, \tag{31}$$

where according to Planck

$$\mathcal{E} = \frac{\hbar\omega}{2} \coth \frac{\hbar\omega}{2k_B T} \tag{32}$$

is the energy of the quantum oscillator in a thermostat. According to SUR

$$(\mathcal{UP}) = J = \frac{\mathcal{E}}{\omega}. \tag{33}$$

In the limit $T \to 0$ the quantity J has the meaningful property: it come up to its minimal value

$$(J)_{min} \equiv J_0 = \frac{\mathcal{E}_0}{\omega} = \frac{\hbar}{2}, \tag{34}$$

where ε_0 is the energy of the oscillator basic state. So J_0 has a fundamental sense of the internal or self action that the object has initially due to the quantum stochastic influence of the "cold" vacuum.

Considering this fact we obtain from SUR

$$J^2 = J_T^2 + J_0^2. \tag{35}$$

Here

$$J_T = \frac{\hbar}{2}\left[\sinh\frac{\hbar\omega}{2k_BT}\right]^{-1} \tag{36}$$

can be interpreted as an effect induced by the thermal stochastic influence of the environment.

Analogically we can rewrite SUR in another form

$$\mathcal{E}^2 = \mathcal{E}_T^2 + \mathcal{E}_0^2, \tag{37}$$

where

$$\mathcal{E}_T = \frac{\hbar\omega}{2}\left[\sinh\frac{\hbar\omega}{2k_BT}\right]^{-1} \tag{38}$$

is the energy induced by a thermal stochastic influence of the environment.

Now let us compare the two formulas (35) and (37) with the formula for full relativistic energy in the Special Relativity Theory (SRT)

$$\mathcal{E}^2 = p^2 + m^2 = \mathcal{E}_p^2 + \mathcal{E}_0^2 \tag{39}$$

(we put the light velocity $c = 1$). Here the quantity $\mathcal{E}_0 \equiv m$ is the self energy, initially belonged to an object and associated with its mass, but $\mathcal{E}_p \equiv p$ is an energy induced by motion of the object and associated with its momentum.

Considering this resemblance we claim that $(\mathcal{E}, \mathcal{E}_p)$ and $(\mathcal{E}, \mathcal{E}_T)$ are two time-like vectors in the corresponding 2-dimensional pseudo-Euclidean spaces. Accordingly $\mathcal{E}_0^2 = m^2$ and $\mathcal{E}_0^2 = (\hbar\omega/2)^2$ are their squared lengths, i.e. invariants.

Now one can realize a new interpretation for the sense of the saturated SUR. We know that the formula (39) is usually considered as a definition of a mass-shell in the pseudo-Euclidean momentum space. This fact corresponds to the characteristics of real particles. But for virtual particles we have the unequality in this formula. It means they exit from the mass- shell.

From this point of view one can claim that the equality in SUR answers the choice of some real states for which the vector $(\mathcal{E}, \mathcal{E}_T)$ is on a certain "frequency shell". Such states are the thermal $\tilde{\mathrm{N}}\tilde{\mathrm{N}}$S describing a thermal equilibrium. All different states have the sense of virtual states for which the same vector is out of the "frequency- shell" or "equilibrium - shell". Probably, they correspond to non-equilibrium.

As some remark we remind that the group Lie SU (1,1) of the Bogoliubov (u, v)-transformations is local isomorphic to the Lorentz-group in 2-dimensional space-time. At the same time there exists an analogy between the pairs of parameters: on the one hand

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}; \quad \beta = \frac{v}{c}, \tag{40}$$

and on the other hand

$$\gamma_T = \coth \frac{\hbar\omega}{2k_B T}; \tag{41}$$

$$\beta_T = \left[\coth \frac{\hbar\omega}{2k_B T}\right]^{-1}. \tag{42}$$

It is not difficult to see that the limit behavior of the corresponding qualities is similar:

at $T \to 0 \quad \beta_T \to 0$ (at $v \to 0 \quad \beta \to 0$) and

at $T \to \infty \quad \beta_T \to 1$ (at $v \to c \quad \beta \to 1$).

VI. Consequences

Summarizing all the results obtained above we can recognize a certain resemblance between properties of the states space in non-classical Physics and the events space in classical Physics. We collect them in Table 1.

One can see that in the absence of thermal influence (at $k_B = 0$ as in QM) or, correspondingly of gravitation (at $G = 0$ as in SRT) there are the simplest Riemannian structures with a diagonal metric and zero curvature. At the same time the role of invariants, limiting the minimal values of corresponding quantities that are possible in Nature, plays either the self-

space	The theories	Metric	Curvature	Structure	
				Riemannian	symplectic
of states	QM ($\hbar \neq 0$, $k_B = 0$)	Nondegenerate, diagonal $f(\hbar)$	0	+	$\frac{1}{2}\hbar$
	QGST ($\hbar \neq 0$, $k_B \neq 0$,)	Nondegenerate, nondiagonal $f(\hbar/2\,k_B)$	$f(\hbar/2\,k_B)$	+	$\frac{1}{2}\hbar$
of events	SRT ($1/c \neq 0$ $G=0$,)	Nondegenerate, diagonal, $f(1/c)$	0	+	$1/c$?
	GRT ($1/c \neq 0$, $G \neq 0$)	Nondegenerate, nondiagonal $f(G/c^2)$	$f(G/c^2)$	+	$1/c$?

Table 1.

action squared - $(\hbar/2)^2$ or the electrodynamics constant squared - $(1/c)^2$. These minimal limitations are initially intrinsic to the object environment in the form of "cold" vacuum.

The presence in the object environment of a matter that is subject to gravitation (at $G \not\equiv 0$ as in General Relativity Theory - GRT) or the change of "cold" vacuum to "warm" one (at $k_B \not\equiv 0$ as in QST) leads to cardinal new properties. First of all, the non-trivial Riemannian structures appear so that the metrics become non-diagonal. Second, in both cases the curvature of space becomes not equal to zero. One can respect these geometrical properties as an indicator of some external effects characterized either by the constant G or by the constant k_B. All these circumstances may be considered as first steps to join both spaces of matter existence.

The last question that arises here is connected with the notion of symplectic structure. In non-classical Physics the modulus of corresponding quantity is an invariant too that is temperature independ and equal to $(\hbar/2)^2$. On this ground we have a reason to say some hypothesis. We can suppose that there is a symplectic structure in the events space. Its invariant characteristic must be a quantity that is equal to $(1/c)^2$. We have the opinion that in the future it will be desirable to modify the description of the events space. On this way the presence of symplectic structure could follow from the fundamental space-time theory.

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SOME GEOMETRICAL PROPERTIES OF THERMAL CORRELATED – COHERENT STATES ‡‡

A. D. Sukhanov[1], **Yu.P. Rybakov**[2], **V. I. Sanyuk**[2]

[1] Joint Institute for Nuclear Research, Dubna, Russia
[2] Russian Peoples Friendship University, Moscow, Russia

Abstract

Some geometrical properties of the generalized states space simultaneously generated by quantum and thermal stochastic influence of environment are studied. As a model we choose a quantum oscillator (QO) in the thermostat. Its states have a sense of thermal correlated - coherent states (TCCS). Earlier we found a wave function in the TCCS that has a temperature dependent amplitude and phase. Under the suitable parametrization it generates a Riemannian structure on the states space. The last circumstance allows us to calculate the Gaussian curvature in the space and to make a comparative study of the geometrical properties for different TCCS in all the temperature range.

CONTENTS

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I. Introduction

Recently it became apparent that our knowledge of matter structure was very approximate. Under these conditions the significance of universal non-model theories like Geometry and Thermodynamics essentially increases. Within the frame of classical (deterministic) Physics the main interest attracts Geometry of events space , i.e. four-dimensional space-time. It is the subject of many papers.

However, in non-classical (stochastic) Physics such as Quantum mechanics (QM), Statistical thermodynamics (ST), and their generalizations Geometry does not appear to be the subject of systematical study. As is well known, the concept of states space is the central one in these theories.

The results of Provost and Vallee [1] and Ruppeiner [2] have shown that in QM and ST one can introduce the Riemannian structure in the corresponding states spaces. Its typical features can be expressed in terms of dispersions of the system random characteristics and their correlators.

In this paper some geometrical properties of the generalized states space simultaneously generated by quantum and thermal stochastic influence of environment are studied. As a model we choose a quantum oscillator (QO) in the thermostat. In another words, QO locates in the thermofield vacuum and its states have a sense of thermal correlated-coherent states (TCCS)[3].

We assume that a wave function in the TCCS has a temperature - dependent amplitude and phase. Under the suitable parameterization it generates a Riemannian structure on the states space. The last circumstance allows us to introduce a gauge-invariant metric tensor and calculate the Gaussian curvature in the space. The latter fact gives us a possibility to make a comparative study of the geometrical properties for different TCCS in the entire temperature range.

II. Geometrical interpretation of the wave function in the TCCS space

The starting point of our study is the Schroedinger uncertainty relation for the variables "coordinate-momentum". In the case of equality it is known as the saturated one. In another words, it has a form of equality:

$$\overline{(\Delta p)^2}\ \overline{(\Delta q)^2} = |\langle|\Delta\widehat{p}\,\Delta\widehat{q}\,|\rangle|^2 = \frac{\hbar^2}{4}\left[\coth\frac{\hbar\omega}{2k_BT}\right]^2. \tag{1}$$

For QO in the thermostat we found [4] the wave function in TCCS satisfying this relation has the form

$$\psi(q) = [2\pi\overline{(\Delta q)^2}]^{-1/4}\exp\left\{-\frac{q^2}{4\overline{(\Delta q)^2}}(1-i\alpha)\right\}, \tag{2}$$

where the coefficient α and the coordinate dispersion are

$$\alpha = \left[\sinh\frac{\hbar\omega}{2k_BT}\right]^{-1}; \tag{3}$$

$$\overline{(\Delta q)^2} = \frac{\hbar}{2m\omega}\coth\frac{\hbar\omega}{2k_BT}. \tag{4}$$

We will interpret $\psi(q)$ as some surface in the Hilbert space of TCCS. To this end we consider the parameters as some effective coordinates in the two-dimensional Riemannian space. The choice of parameters being non-unique, we review only one possible variant.

Let us represent $\psi(q)$ as a ray in the projective Hilbert space putting

$$\psi(q) \equiv \psi(s_1 s_2) = \gamma s_1 \exp\{-\beta q^2(s_1^4 - is_2)\}. \tag{5}$$

Here s_1, s_2 are effective coordinates of the Riemannian space

$$s_1 = \left[\coth\frac{\hbar\omega}{2k_BT}\right]^{-1/4}; \tag{6}$$

$$s_2 = \left[\cosh\frac{\hbar\omega}{2k_BT}\right]^{-1} \tag{7}$$

and constants are

$$\gamma = \left[\frac{\pi\hbar}{m\omega}\right]^{-1/4}; \quad \beta = \frac{m\omega}{2\hbar}. \tag{8}$$

Following [1]let us introduce the gauge-invariant metric tensor

$$g_{ik} = \gamma_{ik} - \beta_i\beta_k, \tag{9}$$

where

$$\gamma_{ik} = \Re\left\langle \frac{\partial\psi^*}{\partial s_i}\middle|\frac{\partial\psi}{\partial s_k}\right\rangle; \tag{10}$$

$$\beta_k = -i\ \langle\psi^*|\frac{\partial\psi}{\partial s_k}\rangle. \tag{11}$$

III. Some geometrical characteristics of TCCS space

Knowing the wave function $\psi(s_1, s_2)$ we can first of all calculate the components of the gauge-invariant metric tensor g_{ik} using the formulas above. Neglecting the details of the calculations we get

$$\beta_1 = 0; \quad \beta_2 = \frac{1}{4}s_1^{-4}; \tag{12}$$

$$g_{11} = \gamma_{11} = 2s_1^{-2}; \tag{13}$$

$$g_{12} = g_{21} = \gamma_{12} = 0; \tag{14}$$

$$\gamma_{22} = \frac{3}{16}\ s_1^{-8}; \tag{15}$$

$$g_{22} = \gamma_{22} - \beta_2^2 = \frac{1}{8}\ s_1^{-8}. \tag{16}$$

Now we calculate the Riemannian metric on the studied surface in the TCCS space

$$dl^2 = g_{ik}(s_i, s_k)ds_ids_k, \tag{17}$$

where dl stands for the elementary length of a curve on the surface $\psi(s_1, s_2)$.

At last the determinant of the metric tensor g_{ik} is determined by the formula

$$g = \frac{1}{4}\ s_1^{-10}, \tag{18}$$

where the tensor g is a diagonal one.

Correspondingly we can get the characteristics of symplectic structure

$$\sigma_{ik} = -\sigma_{ki} = \Im \left\langle \frac{\partial \psi^*}{\partial s_i} \middle| \frac{\partial \psi}{\partial s_k} \right\rangle \tag{19}$$

In the given case

$$\sigma_{12} = -\sigma_{21} = -\frac{1}{2}\, s_1^{-5} \tag{20}$$

It is interesting that

$$g = g_{11} g_{22} = |\sigma_{12}|^2 \tag{21}$$

Note that the main property of all these quantities is the dependence on the parameter $(s_1)^{-1}$ in the form (6).

One can note that under the temperature variation in the range $0<T<\infty$ the parameter $(s_1)^{-1}$, where s_1 has the form (6), takes values in the range

$$1 \leqslant s_1^{-1} \leqslant \left(\frac{2k_B T}{\hbar\omega}\right)^{1/4}. \tag{22}$$

We underline that the dependence on the temperature is the significant peculiarity of the TCCS space. It should be also recalled that the coordinate dispersion (4) so the quantities g_{11}, g_{22}, and σ_{12} can be expressed through the coordinate dispersion of QO as follows

$$s_1^{-4} = \frac{2m\omega}{\hbar}\ \overline{(\Delta q)^2}. \tag{23}$$

Thus we establish the relation between features of the TCCS space and fluctuations of physical characteristics in non-classical Physics.

IV. Connection coefficients and the Riemann-Christoffel tensor

To calculate a curvature in 2-dimensional Riemannian space under the chosen parameterization let us first calculate connection coefficients. Because the metric is non-degenerate, i.e. *det* $g_{ij} \not\equiv 0$, there exists a unique connection that is symmetric and consistent with metric g_{ij}.

It is defined as follows [5,6]:

$$\Gamma_{ij,k} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial s_i} + \frac{\partial g_{ik}}{\partial s_j} - \frac{\partial g_{ij}}{\partial s_k}\right). \tag{24}$$

In our case only three connection coefficients are not trivial:

$$\Gamma_{11,1} = \frac{1}{2}\frac{\partial g_{11}}{\partial s_1} = 2s_1^{-3}; \tag{25}$$

$$\Gamma_{22,1} = -\frac{1}{2}\frac{\partial g_{22}}{\partial s_1} = \frac{1}{2}s_1^{-9}; \tag{26}$$

$$\Gamma_{12,2} = \frac{1}{2}\frac{\partial g_{22}}{\partial s_1} = -\frac{1}{2}s_1^{-9}. \tag{27}$$

They depend on the temperature through the expressions (6) for coordinate s_1 of effective Riemannian space and (4) for coordinate dispersion QO in the thermostat.

Knowing the quantities $\Gamma_{ij,k}$ one can calculate components of the Riemann-Christoffel tensor

$$R_{mlkj} = g_{mi}R^i_{lkj} = g_{mi}\left(\frac{\partial \Gamma^i_{lj}}{\partial s_k} - \frac{\partial \Gamma^i_{lk}}{\partial s_j} + \Gamma^i_{nk}\Gamma^n_{lj} - \Gamma^i_{nj}\Gamma^n_{lk}\right), \tag{28}$$

where it is necessary to take into account the expression

$$\Gamma_{i,kl} = g_{im}\Gamma^m_{kl}. \tag{29}$$

In the 2-dimensional case from the symmetry of the Riemann-Christoffel tensor follows that its unique component is

$$R_{2112} = -\frac{\partial \Gamma_{12,2}}{\partial s_1} - \Gamma_{11,i}\Gamma_{22,j}g^{ij} + \Gamma_{12,i}\Gamma_{12,j}g^{ij}. \tag{30}$$

If one inserts the obtained quantities in this formula one gets

$$R_{2112} = -\frac{\partial \Gamma_{12,2}}{\partial s_1} - \Gamma_{11,1}\Gamma_{22,1}\, g^{11} + \Gamma_{12,2}\Gamma_{12,2}\, g^{22}. \tag{31}$$

Taking into account that $g^{11} = 1/g_{11}$ and $g^{22} = 1/g_{22}$ one can finally obtain

$$R_{2112} = -\frac{9}{2}\, s_1^{-10} + \frac{1}{2}\, s_1^{-10} + 2s_1^{-10} = -2s_1^{-10}, \tag{32}$$

where the dependence on temperature appears again through coordinate dispersion.

Finally, from the formulas above one can calculate the Gaussian scalar curvature K:

$$K = \frac{R_{2112}}{g} = -8. \tag{33}$$

It should be underlined that the curvature of the TCCS space at the point associated with the normalized state in the projective Hilbert space is constant and negative. This metric corresponds to geometrical features of hyperboloid.

V. Conclusion

Let us summarize our geometric results and make some physical comments.

If we fix some set of the wave function characteristics for QO in thermostat, we can use its geometrical interpretation as a surface in the TCCS space.

In this case such features of the space as metric tensor components, connection coefficients, and components of the Riemann-Christoffel tensor depend on coordinate dispersion associated with the thermostat temperature.

The Gaussian scalar curvature of the surface associated with the wave function is constant and negative. We claim that this result shows resemblance to that obtained earlier [1] for the same case by group-theoretical method.

We hope that our results will be useful for establishing of similarity between the properties of events space in classical and those of the states space in non-classical Physics.

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GEOMETRICAL MODEL WITH TWO EXPONENTS FOR DESCRIBING THE PROTON-PROTON SCATTERING AT HIGH ENERGIES

Z. Tarics

Institute of Electron Physics, Ukrainian National Academy of Sciences, Uzhgorod, Ukraine

Abstract

A dipole model of pomeron with two independent exponents is suggested. It is shown that the appearance of the minima and maxima observed experimentally in the differential cross-sections of elastic pp-scattering at high energies could be described within the framework of the above model. The model is analyzed; the limitations for its certain parameters are obtained.

I. Introduction

The experimental differential cross-sections of elastic pp-scattering for momentum transfer $0.5 < |t| < 14$ GeV^2 (energy $\sqrt{s} = 23.5 - 62$ GeV) demonstrate different minima and maxima. These extrema shift slowly to lower $|t|$ with increasing energy. In [1-3], an elegant model was proposed and the above behavior of cross-sections has been satisfactorily described. The advantage of the model is its simplicity (it includes only the dipole pomeron) and a small number of parameters (four), which can be fitted from experiments. However, its shortcoming is that it leads to a decreasing ratio of σ_{el}/σ_{tot}, which tends to an asymptotic constant value.

The model [3, 4] includes a triple pomeron and, therefore, it breaks the unitarity. However, due to a large number of parameters (10-20), this model describes perfectly the above minima, maxima and the σ_{el}/σ_{tot} ratio.

Here a dipole pomeron model is suggested containing two exponential terms dependent on t. In this model, the minima and the maxima of the differential cross-sections for the elastic pp-scattering appear as well. In addition, it imposes several restrictions on the parameters and allows one to determine those values of $\sqrt{s}$, for which the extrema would appear or disappear.

II. Dipole model with one exponent

An ansatz for the dipole pomeron amplitude has a geometrical form [2,3]

$$u(s,t) = isg_0 \left(c_1 R_1^2 \exp(R_1^2 t) + c_2 R_2^2 \exp(R_2^2 t)\right),$$

where radii $R_{1,2}$ depend on the energy; g_0, c_1 and c_2 are constants. Making known transformations of this amplitude, performing some substitutions and choosing definitely the constants c_1, c_2 [2], one obtains [3]

$$T(s,t) = A\left(-i\frac{s}{s_0}\right)^{\alpha(t)} \left\{\left[1+\frac{1}{b}\ln\left(-i\frac{s}{s_0}\right)\right] e^{b[\alpha(t)-1]} - \gamma\ln\left(-i\frac{s}{s_0}\right)\right\}, \qquad (1)$$

where A, b, γ are the constants or parameters. s_0 can be chosen as a dimensionality parameter: $s_0 = 1$ GeV^2. The pomeron trajectory was chosen in a linear form:

$$\alpha(t) = 1 + \alpha' t. \qquad (2)$$

In this model, simple formulae were obtained [2] for the positions of the minima and maxima as well as for their behavior.

III. Dipole model with two exponents

The amplitude with two exponents for pp ($\bar{p}\bar{p}$)-scattering is also chosen in a simple form:

$$P(s,t) = isg_0\left[e^{at} + ce^{bt}\ln\left(-i\frac{s}{s_0}\right)\right]\left(-i\frac{s}{s_0}\right)^{\alpha(t)-1}, \qquad (3)$$

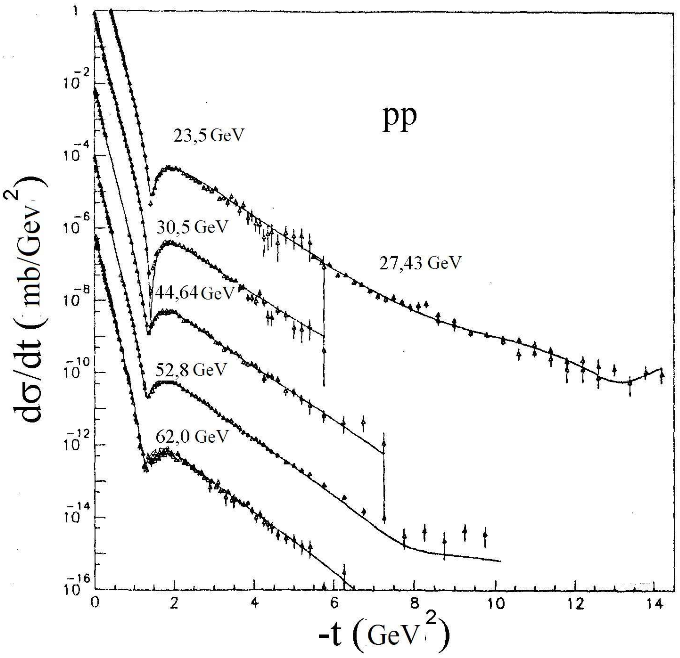

Figure 1: Differential cross-sections of elastic pp-scattering.

where g_0, a, b, c are the constants and $s_0 = 1$ GeV^2.

Here the amplitude (3) is normalized in such a way that the differential and total cross-sections could be calculated by the following formulae:

$$\frac{d\sigma_{el}}{dt} = \frac{\pi}{s^2}|P(s,t)|^2, \tag{4}$$

$$\sigma_{tot} = \frac{4\pi}{s}\mathrm{Im}P(s,t=0). \tag{5}$$

For the elastic differential cross-section we obtain an expression:

$$\frac{d\sigma_{el}}{dt} = \frac{\pi g^2}{s^2}\left[\left(e^{at} + ce^{bt}\ln\frac{s}{s_0}\right)^2 + \left(\frac{\pi c}{2}\right)^2 e^{2bt}\right]\left(\frac{s}{s_0}\right)^{2\alpha(t)}, \tag{6}$$

where substitution $g = g_0 s_0$ was made. The total cross-section is given by the following formula:

$$\sigma_{tot} = -\frac{4\pi g}{s_0}\left(1 + c\ln\frac{s}{s_0}\right). \tag{7}$$

IV. Extrema in differential cross-sections

Here we also choose a linear pomeron trajectory (2). Let us find the extrema of the differential cross-section (6) on t. The extremal points should be obtained from the following equation:

$$(a+\alpha' L)e^{2at} + cL(a+b+2\alpha' L)e^{(a+b)t} + \left[c^2(b+\alpha' L)\left(L^2 + \frac{\pi^2}{4}\right)\right]e^{2bt} = 0, \tag{8}$$

where $L = \ln(s/s_0)$. Multiplying this equation by e^{-2bt} and introducing a new variable

$$x = e^{(a-b)t}, \tag{9}$$

we obtain

$$(a+\alpha' L)x^2 + cL(a+b+2\alpha' L)x + c^2(b+\alpha' L)\left(L^2 + \frac{\pi^2}{4}\right) = 0. \tag{10}$$

The solutions are

$$x_\pm = -\frac{c}{2(a+\alpha' L)}\left[L(a+b+2\alpha' L) \pm \sqrt{L^2(a-b)^2 - \pi^2(a+\alpha' L)(b+\alpha' L)}\right]. \tag{11}$$

Let us analyze formula (11). First note that for the pomeron trajectory slope we chose here $\alpha' = 0.25$ GeV^{-2} and, from experiment, $c > 0$. With this α' accepted in the most of papers a large number of experimental data were described not only for pp- and $p\bar{p}$-scattering but also for other high-energy processes. We notice that our amplitude reflects the situation when the pomeron gives a main contribution to the physical values, which characterize the processes. This statement is correct beginning from the energies $\sqrt{s} \sim 4 - 5$ GeV. Thus, $L \geq 0$.

One can see from expression (9) that the physical values of x are positive. On the other hand, to let the minima and the maxima occur in the differential cross-section, the determinant of equation (10) must be positive. We require for all $L \geq 0$

$$L^2(a-b)^2 - \pi^2(a+\alpha' L)(b+\alpha' L) > 0. \qquad (12)$$

The left-hand part of inequality (12) could be considered a positively defined function of L:

$$f(L) = [(a-b)^2 - (\pi\alpha')^2]L^2 - \pi^2\alpha'(a+b)L - \pi^2 ab > 0. \qquad (13)$$

It is obvious that

$$(a-b)^2 - (\pi\alpha')^2 > 0. \qquad (14)$$

Inequality (13) must be valid for $L = 0$ as well. Thus,

$$-\pi^2 ab > 0. \qquad (15)$$

Not restricting the generality, we can choose $a > 0$ and $b < 0$. Then it follows from (14) that

$$a+b > 0, \qquad a > |b|. \qquad (16)$$

The function $f(L)$ has a minimum at

$$L_{min} = \frac{\pi^2\alpha'(a+b)}{2[(a-b)^2 - (\pi\alpha')^2]}, \qquad (17)$$

i.e., in fact, $L_{min} > 0$. It is obvious that the minimal value is

$$f(L_{min}) = -\pi^2 ab \frac{\pi^2\alpha'^2(a+b)^2}{2[(a-b)^2 - (\pi\alpha')^2]} > 0. \qquad (18)$$

This inequality may be represented in a following form

$$a|b| - \frac{\pi^2\alpha'^2(a-|b|)^2}{4[(a+|b|)^2 - (\pi\alpha')^2]} > 0. \qquad (19)$$

Hence, it follows

$$(a+|b|)^2(4a|b| - \pi^2\alpha'^2) > 0, \qquad (20)$$

i.e.

$$4a|b| > (\pi\alpha')^2, \qquad (21)$$

or

$$a|b| > (\pi\alpha'/2)^2 = 0.15. \tag{22}$$

In the physical region, $t \leq 0$ and $0 < x \leq 1$. From formula (11) we obtain for $L = 0$

$$x_0 = -\frac{c}{2a}\left(\pm\sqrt{\pi^2 a|b|}\right) \tag{23}$$

i.e. the physical solution is obtained by choosing the minus sign. Thus,

$$\frac{\pi c}{2a}\sqrt{a|b|} \leq 1 \tag{24}$$

and

$$|b| \leq \frac{4a}{(\pi c)^2}. \tag{25}$$

So, the constant b satisfies the following inequalities

$$\left(\frac{\pi\alpha'}{2\sqrt{a}}\right)^2 < |b| \leq \frac{4a}{(\pi c)^2}. \tag{26}$$

From this expression the lower limit for the constant a results:

$$a > \frac{\pi^2\alpha'|c|}{4}. \tag{27}$$

It is seen from (22) and (11) that for some $L = L_1$ the value of x_- becomes zero and for $L > L_1$ the solution is $x_- < 0$. The equation for L_1 has a form

$$(4L_1^2 + \pi^2)(a + \alpha' L_1)(b + \alpha' L_1) = 0, \tag{28}$$

and from here we obtain

$$L_1 = \frac{|b|}{\alpha'}. \tag{29}$$

Thus, for $L \geq L_1$ the extrema vanish. This fact confirms the data for $\bar{p}p$-scattering obtained up to $\sqrt{s} = 546$ GeV.

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THE FRENET APPARATUS OF NULL CURVE AND THE NULL HELIX IN R_1^{M+2}

A. Altın

Hacettepe University, Beytepe Ankara Turkey

Abstract

In this work we calculated the Frenet apparatus of a null curve C in R_1^{m+2} in terms of the Frenet apparatus of the curve C^* which is the orthogonal projection of C on R^{m+1}. We also give the theorems which provides some information about a null helix. If C is a null helix then it must be contained in a four dimensional subspace in R_1^{m+2}.

I. Preliminaries

The smooth curve $C = \alpha(I)$ in a semi-Riemannian manifold (M^{m+2}, g) is said to be a null curve if the velocity vector to C at any point is a null vector.

Let TC be the tangent bundle of C and $TC^{\perp} = \cup_{t\in I} T_{\alpha(t)}C^{\perp}$, where $T_{\alpha(t)}C^{\perp} = \{V_{\alpha(t)} \in T_{\alpha(t)}M : g(V_{\alpha(t)}, \alpha'(t)) = 0\}$.

At each point $\alpha(t)$, we choose a complementary vector space to $T_{\alpha(t)}C$ in $T_{\alpha(t)}C^{\perp}$. Denote by $S(T_{\alpha(t)}C^{\perp})$, this chosen subspace. Hence, we get a vector bundle $S(TC^{\perp})$ on α. Since

$$TC^{\perp} = TC \perp S(TC^{\perp}),$$

$S(TC^{\perp})$ is a vector bundle of rank m. The non-degenerate vector bundle $S(TC^{\perp})$ is called a screen vector bundle of C. Therefore we have

$$TM \mid_C = S(TC^{\perp}) \perp S(TC^{\perp})^{\perp} \tag{1}$$

where $S(TC^{\perp})^{\perp}$ is a complementary orthogonal vector bundle to $S(TC^{\perp})$ in $TM\mid_C$.

Theorem 1.1. Let C be a null curve of a proper semi-Riemannian manifold (M,g) and $S(TC^{\perp})$ be a screen vector bundle of C. Then there exists a unique vector bundle $ntr(C)$ over C of rank 1, such that on each coordinate neighbourhood $U \subset C$ there is a unique $N \in \Gamma(ntr(C)\mid_U)$ satisfying

$$g(\alpha'(t), N) = 1, \quad g(N,N) = 0, \quad g(N,X) = 0, \quad \forall X \in \Gamma(S(TC^{\perp})\mid_U. \quad (2)$$

Consider

$$tr(C) = ntr(C) \perp S(TC^{\perp}),$$

from (1), (2) then we have the following sum

$$TM\mid_C = TC \oplus tr(C) = (TC \oplus ntr(C)) \perp S(TC^{\perp}). \quad (3)$$

The vector field N, which was constructed in this theorem, is said to be the null transversal vector field of C with respect to α' [3.p.53]. A null curve C in R_1^{m+2} is given locally by the equation of the following form

$$\alpha(s) = (s, \quad \int_0^s \alpha_1, ..., \int_0^s \alpha_{m+1}), \quad (4)$$

where, $\alpha_1 = cosb_1(s)ds + c_1$, $\alpha_a = cosb_a(s)\prod_{k=1}^{a-1} sinb_k(s)ds + c_a$, $a \in \{2,...,m\}$, $\alpha_{m+1} = \prod_{k=1}^{m} sinb_k(s)ds + c_{m+1}$, $c_k \in R$, b_k are smooth functions for any $k \in \{1,...,m\}$, and s is the arc-length of the orthogonal projection $C^* = \alpha^*(I)$ of C on R^{m+1} give by, [3,p.73],

$$\alpha^*(t) = \{\alpha_1^*(t), \alpha_2^*(t), ..., \alpha_{m+1}^*(t)\}.$$

In this paper, we mean by α^{l}'s, $(1 \leq l \leq r)$ the derivatives of the curve α.

Let α^* be a regular curve in R^{m+1} and $\psi = \{(\alpha^*)'(t), (\alpha^*)''(t), ..., \alpha^{*r}(t)\}$ be a maximal linearly independent set. The orthonormal system $\{V_1(t), V_2(t), \ldots, V_r(t)\}$ can be obtained from ψ. This is called a Frenet frame at the point $\alpha^*(t)$, [4].

Defnition 1.2. Let α^* be a regular curve in R^{m+1} and $\{V_1(t), V_2(t), \ldots, V_r(t)\}$ be the Frenet frame at the point $\alpha^*(t)$.

The functions $k_i : I \longrightarrow R$ defined by

$$k_i(t) = g(V_i'(t), V_{i+1}(t)), \quad 1 \leq i \leq r-1 \tag{5}$$

are called curvature functions on α^*. Moreover, the real number $k_i(t)$ is called the $i-th$ curvature on α^* at the point $\alpha^*(t)$.

Theorem 1.3. Let α^* be a unit speed curve in R^{m+1} and the set $\{V_1(t), V_2(t), ..., V_r(t)\}$ be the Frenet frame at the point $\alpha^*(t)$. Then, the followings hold, [7,p.194],

$$\begin{aligned} V_1'(t) &= k_1(t)V_2(t), && (6)\\ V_i'(t) &= -k_{i-1}(t)V_{i-1}(t) + k_i(t)V_{i+1}(t), \quad 1 < i < r, && (7)\\ V_r'(t) &= -k_{r-1}(t)V_{r-1}(t). && (8) \end{aligned}$$

2. Helix

Theorem 2.1. Let α and α^* be the curves as in equation (4) and $\{V_1, V_2, ..., V_r\}$, $\{k_1, k_2, ..., k_{r-1}\}$ be Frenet fields and curvature functions of curve α^* respectively. Then, α is a null curve in R_1^{r+1}. More over if we choose $S(TC^{\perp})$ spanned by $\{W_2, ..., W_r\}$, then we have the null transversal vector field $N = \frac{1}{2}(-1, V_1)$, and the Frenet equations are

$$\begin{aligned} \alpha'' &= k_1 W_2\\ N' &= \frac{1}{2}k_1 W_2\\ (W_2)' &= -\frac{1}{2}k_1\alpha' - k_1 N + k_2 W_3\\ (W_3)' &= -k_2 W_2 + k_3 W_4\\ (W_4)' &= -k_3 W_3 + k_4 W_5\\ &\quad \ldots\ldots\ldots\ldots\ldots\ldots\\ (W_{r-1})' &= -k_{r-2}W_{r-2} + k_{r-1}W_r\\ (W_r)' &= -k_{r-1}W_{r-1}, \end{aligned} \tag{9}$$

where

$$W_j = (0, V_j) \quad j \in \{2, ..., r\} \tag{10}$$

and the Frenet frame is $F = \{\alpha', N, W_2, ..., W_r\}$ on R_1^{r+1} along α.

Proof. From (4) we have

$$\alpha' = (1, (\alpha^*)'), \quad \alpha^j = (0, (\alpha *)^j), \quad j \geq 2.$$

Therefore $\{\alpha', \alpha'', ..., \alpha^r\}$, $\ r \leq m+1$ is the maximal linearly independent set. Since α^* has at most (r-1) non zero curvatures, α^* is contained in R^r. Therefore α is also contained in R_1^r.

We choose $S(TC^{\perp}) = span\{W_2, ..., W_r\}$ and a complementary vector bundle H to TC in $S(TC^{\perp})$,

$Y = (0, (\alpha^*)') \in \Gamma(H\mid_U)$ and $g(\alpha', Y) \not\equiv 0$ on U. We calculated the null transversal vector field and found that

$$N = \frac{1}{g(\alpha', Y)}\{Y - \frac{g(Y,Y)}{2g(\alpha', Y)}\alpha'\} = \frac{1}{2}(-1, (\alpha^*)').$$

Since $(\alpha^*)' = V_1$, we have

$$\alpha' = (1, V_1), \qquad N = \frac{1}{2}(-1, V_1). \tag{11}$$

We differentiate (10), (11) and by using (6), (7), (8) we obtain (9).

By the ideas in $[5, p.73]$ and $[6, p.160]$, a helix is defined as a curve which has a constant scalar product of its tangent vector field and a constant vector field.

We now give the definition of a null helix in semi-Euclidean space R_1^{m+2} in a similar way to [2], as follows.

Definition 2.2. Let α be a null curve in R_1^{m+2} and X be a non zero constant vector field. If

$$g(\alpha'(t), X) = \text{constant} \not\equiv 0, \quad \text{for all} \quad t \in I,$$

then, α is said to be a null helix in R_1^{m+2} and *span* $\{X\}$ is said to be the inclination axes of α,

Example 2.3.

$\alpha : R \longrightarrow R_1^3$ be the curve difened by

$$\alpha(t) = (\frac{4}{3}t^3 + t, \quad 2t^2, \quad \frac{4}{3}t^3 - t), \quad X = (1, 0, 1)$$

Example 2.4. Let $a, \sigma, \rho, \omega, d$ be non-zero constants, b be constant and let $\alpha : R \longrightarrow R_1^5$ be the curve difened by

$$\alpha(t) = (at + b, \ \frac{1}{\rho}\sigma cos\rho t, \ \frac{1}{\rho}\sigma sin\rho t, \ \frac{1}{\omega}dcos\omega t, \ \frac{1}{\omega}dcos\omega t), \ X = (1, 0, 0, 0, 0)$$

where $a^2 = \sigma^2 + d^2$.

Definition 2.5. Suppose that $k_1, k_2, ..., k_{n-1}$ are curvature functions of a curve α. A function $H_i : I \longrightarrow R$ defined by

$$H_i(t) = \begin{cases} \frac{k_1(t)}{k_2(t)}, & \text{if} \quad i = 1 \\ \frac{1}{k_{i+1}(t)}\{H'_{i-1}(t) + k_i(t)H_{i-2}(t)\}, & \text{if} \quad 2 \leq i \leq n-2 \end{cases} \tag{12}$$

is called the $i - th$ harmonic curvature function of α.

Lemma 2.6. Let α be a null helix in R_1^{m+2}, $span\{X\}$ be the inclination axes, $\{\alpha', N, W_2, ..., W_{r+2}\}$ be the Frenet frame fields of α and let Π be the orthogonal projection of R_1^{m+2} onto the space $span\{\alpha', N, W_2, ..., W_{r+2}\}$. If $r < m$, then $span\{\Pi(X)\}$ is the inclination axes in $span\{\alpha', N, W_2, \ldots, W_{r+2}\}$.

Proof. We can choose $\{\alpha', N, W_2, ..., W_{r+2}, w_1, ..., w_{m-(r+2)}\}$ as an quasi-orthonormal basis of R_1^{m+2}. In this case

$$\begin{aligned} X &= x_0\alpha' + x_1N + \sum_{i=2}^{r+2} x_iW_i + \sum_{j=1}^{m-(r+2)} b_jw_j, \\ \Pi(X) &= x_0\alpha' + x_1N + \sum_{i=2}^{r+2} x_iW_i. \end{aligned}$$

Since $g(\sum_{j=1}^{m-(r+2)} b_jw_j, \alpha') = 0$ and $g(X, \alpha') = \text{constant} \not\equiv 0$, we have

$$g(x_0\alpha' + x_1N + \sum_{i=2}^{r+2} x_iW_i, \alpha') = \text{constant} \not\equiv 0$$

Since $span\{X\}$ is inclination axes, then $\Pi(X)$ is also non zero and constant.

Theorem 2.7. Let α be a curve in R_1^{m+2} with the Frenet frame field $\{\alpha', N, W_2, ..., W_r\}$ and with harmonic curvatures $H_1, H_2, ..., H_{r-2}$, $r \leq m$. Then, α is a null helix in R_1^{m+2} if and only if H_i 's are constant and $x_1 \not\equiv 0$.

Theorem 2.8. There is a relation between curvatures and harmonic curvatures of a curve in R_1^{m+2} as follows.

$$k_r = \frac{(\sum_{i=1}^{r-2} H_i^2)'}{2H_{r-1}H_{r-2}}, \quad 3 \leq r \leq m-1 \tag{13}$$

Consequently, combining (12), (13) and theorem 2.7 we can give our main theorem.

Theorem 2.9. The curve α is a null helix in R_1^{m+2} if and only if and $k_j = 0$ for $j \geq 3$.

As a consequence of this theorem we obtain the following.

Corollary 2.10. if α is null helix then α is contained in a four dimensional subspaces in R_1^{m+2}.

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ON ISOMETRIC IMMERSIONS OF N-DIMENSIONAL LOBACHEVSKY SPACE INTO (2N-1)-DIMENSIONAL EUCLIDEAN SPACE

Y. Aminov

B.I. Verkin Institute for Low Temperature
Physics and Engineering of NAS of Ukraine, Kharkiv

Abstract

In this work some theorems about isometric immersions of the Lobachevsky space into Euclidean space are presented.

I. Introduction

The study of isometric immersions of n-dimensional Lobachevsky space L^n into Euclidean space E^{2n-1} from the local and global points of view is considered in the author's papers [1] - [10] . In this direction for $n > 2$ there exist also the works by E.Cartan, A.Liber, J.D.Moore, K.Tenenblat, C.-L.Terng, F.Xavier and others.

Let F^n be a regular submanifold in E^{2n-1} isometric to some simple connected domain of the Lobachevsky space L^n with a curvature equal to -1. In terms of curvature coordinates the metric form of F^n can be written in the form

$$ds^2 = \sum_{i=1}^{n} \sin^2 \sigma_i (du^i)^2, \quad \sum_{i=1}^{n} \sin^2 \sigma_i = 1. \tag{1}$$

The functions σ_i satisfy some system of nonlinear differential equations.

For convenience we shall use the following notation:

$$H_i = \sin\sigma_i, \quad \beta_{ij} = \frac{1}{H_i}\frac{\partial H_j}{\partial u_i}, \quad i \neq j.$$

For a regular immersion $H_i > 0$. Then the following system of differential equations describes the isometric immersions of the Lobachevsky space L^n into E^{2n-1}

$$\frac{\partial H_j}{\partial u_i} = \beta_{ij}H_i, \quad \frac{\partial \beta_{ij}}{\partial u_j} + \frac{\partial \beta_{ji}}{\partial u_i} + \sum_q \beta_{iq}\beta_{jq} = 0,$$

$$\frac{\partial H_i}{\partial u_i} = -\sum_q \beta_{iq}H_q, \quad \frac{\partial \beta_{ij}}{\partial u_i} + \frac{\partial \beta_{ji}}{\partial u_j} + \sum_q \beta_{qi}\beta_{qj} = H_iH_j,$$

$$\frac{\partial \beta_{ij}}{\partial u_k} = \beta_{ik}\beta_{kj}, \quad \textit{where} \quad i \neq j \neq k \neq i.$$

This system is completely integrable and is a generalization of well-known "sin-Gordon" equation.

It is natural to call it the system "Lobachevsky-Euclid" or briefly "LE-system". The solution of this system depends on $n(n-1)$ analytical functions of one variable.

II. Section

On the Grassmann image of an immersion

Let N^k be some k-dimensional subspace in E^{n+k} through the fixed point O. Let $e_1, ..., e_{n+k}$ be a fixed orthonormal frame in E^{n+k}. We take in N^k some orthonormal frame, which consists of unit vectors $\xi_1, ..., \xi_k$ and let ξ_i^j be the coordinates of ξ_i with respect to $e_1, ..., e_{n+k}$. We call the following quantities the Plücker coordinates of N^k

$$p^{i_1 \dots i_k} = \begin{vmatrix} \xi_1^{i_1} & \dots & \xi_1^{i_k} \\ \dots & \dots & \dots \\ \xi_k^{i_1} & \dots & \xi_k^{i_k} \end{vmatrix}.$$

Plücker coordinates $p^{i_1 \dots i_k}$ are components of the simple polyvector $p = [\xi_1, ..., \xi_k]$ generated by the vectors $\xi_1, ..., \xi_k$. Well ordered set of these components with condition $i_1 < i_2 ... < i_k$ gives us a point P in the Euclidean space E^m, where $m = C^k_{n+k}$. Since we consider the Grassmann manifold $G_{k,n+k}$ as some submanifold of E^m we can introduce to $G_{k,n+k}$ a metric $d\sigma^2$, which is induced by ambient space E^m

$$d\sigma^2 = \sum_{i_1,\dots<i_k} (dp^{i_1 \dots i_k})^2.$$

Let F^n be a regular submanifold in the Euclidean space E^{n+k} with the position vector $r = r(u^1, ..., u^n)$ and curvilinear coordinates $u^1, ..., u^n$.Grassmann map $\psi : F^n \to G_{n-1,2n-1}$ correlates the (n-1)-dimensional space N passing through some fixed point $O \in E^{2n-1}$ with every point $x \in F^n$, the space N being parallel to normal space N_x of F^n at the point x (i.e. it corresponds to every point x some point of Grassmann manifold $G_{n-1,2n-1}$). The image of this map $\psi(F^n)$ we denote Γ^n. The Grassmann mapping ψ transfers coordinates from F^n onto the image Γ^n. So we can write the position vector of a point of Γ as a vector-function

$$p = p(u^1, ..., u^n).$$

By using of the Weingarten decomposition we can obtain for the metric dl^2 of the Grassmann image Γ^n the following expression

$$dl^2 = dp^2 = \sum_{\alpha=1}^{k} L^\alpha_{il} L^\alpha_{js} g^{ls} du^i du^j,$$

where L^α_{il} are the coefficients of the second fundamental form of F^n with respect to its normal vector ξ_α. If F^n is a regular immersion of some domain of the Lobachevsky space L^n into E^{2n-1} and $u^1, ..., u^n$ are curvature coordinates, so in these coordinates the metric of Γ^n is written as follows [2]

$$dl^2 = \sum_{i=1}^{n} \cos^2 \sigma_i (du^i)^2, \quad \sum_{i=1}^{2} \cos^2 \sigma_i = n - 1. \tag{2}$$

From here we obtain

The sum of the metric of a Lobachevsky space and its Grassmann image is the flat metric

$$ds^2 + dl^2 = (du^1)^2 + ... + (du^n)^2.$$

From the expression of dl^2 it follows also that Γ^n is a regular n-dimensional submanifold. The map for $n > 2$ increases the volume of any domain of F^n and the length of any asymptotic line.

It is well known that does not exist local isometric immersions of a n-dimensional Riemannian space with negative curvature into E^{2n-2}. Multidimensional analogy of the pseudosphere is an example of isometric immersion of domain of the Lobachevsky space L^n into E^{2n-1}.

Remember, that for $n = 2$ it has place the Hilbert theorem about nonexistence isometric immersion of complete Lobachevsky plane into E^3. Multidimensional analogy of this theorem is open question. We can give answer only under some additional conditions.

The properties of the Grassmann image imply the following result

Theorem 1. *If the Grassmann image Γ^n lies on a closed n-dimensional manifold and if the Grassmann map is finite-to-one, then the immersion of the full space L^n in E^{2n-1} has singularities.*

It is interesting to investigate different classes of immersions. One of such classes for $n = 3$ arises on condition that the Garssmann image is hyperplanar, i.e. $\Gamma^3 \subset E^9$.(In the general case $\Gamma^3 \subset E^{10}$.) In this case the Plücker coordinates of points of the Grassmann image satisfy the linear equation

$$\sum_{i<j} a_{ij} p^{ij} + \alpha = 0,$$

where a_{ij} and α are constant numbers.

The existence of local isometric immersions with the hyperplanar Grassmann image is proved. In this case the author found the *connection of the theory of isometric immersion with the theory of rigid body rotation with a fixed point in the central field of gravity and the Newton Law of gravity* [3].

We show that the set of equations for isometric immersion of L^3 into E^5 in this case has as subset the Kirchhoff equations

$$\frac{dH}{dt} = [FH], \quad \frac{dF_i}{dt} = C_i(F_jF_k - \epsilon H_jH_k), \quad i,j,k \neq$$

where $H = \{H_i\}$, $F = \{F_i\}$ are 3-dimensional vectors, C_i and ϵ are constant,[ ] is the vector product in E^3. We obtain some number of the first integrals. From existence of these integrals it follows

Every solution H of the system for isometric immersions of L^3 into E^5 with hyperpflat Grassmann image in general case is definite and analytical over all parameter space u^1, u^2, u^3.

This statement does not guarantee that a corresponding immersion of complete space L^3 into E^5 is regular because there the points of $H_i = 0$ and $H_i < 0$ may occur.

In some subcases the theorems about nonimmersion of full Lobachevsky space are proved.

The following question arises in a natural way: can the metric of the Grassmann image have a constant curvature ? The answer to this question for $n = 3$ is given in [6].

Theorem 2. *There is no local C^3 isometric immersion of L^3 into E^5 with constant curvature of the metric of the Grassmann image.*

III. Section

On a family of submanifolds with a constant negative curvature

In [10] we consider a $(n-1)$- parametric family of submanifolds F^n in E^{2n-1} with a constant negative curvature $K_0(F^n)$ in a ball D of the Euclidean space E^{2n-1}. We suppose that this family is included in some (2n-1)-orthogonal coordinate system $u_1, ..., u_{2n-1}$ as a family of coordinate submanifolds $u_{n+1} = const, ..., u_{2n-1} = const$. The author calls this system the Bianchi system of coordinates, if the first n coefficients H_i^2 of the metric form of the ambient space satisfy the following condition

$$\sum_{i=1}^{n} H_i^2 = 1. \tag{3}$$

Bianchi shows for $n = 2$ that the condition (3) satisfied automatically. The author has proved for $n = 3$ that for proving the next theorem it will suffice to demand the condition (3) only on two coordinate curves u_4, u_5 going through the center of the ball D. Besides, remark that on each submanifold F^n one can introduce the curvature coordinates, for which the condition (3) is true. In the paper [10] proved is

Theorem 3. *Assume that a ball of radius ρ in the Euclidean space E^{2n-1} carries a regular Bianchi system of coordinates such that $K_0(F^n) \leq -1$. Then*

$$\rho \leq \frac{\pi}{4}.$$

There exists an example of a regular Bianchi system in a ball $D \subset E^3$ with radius $\rho = \frac{1}{2}$.

As F^n is the submanifold with the flat normal connection, then on F^n there exists a field ξ of normal unit vectors parallel translated in the normal bundle. With the help of this field ξ we construct a map $\varphi : F^n \to S^{2n-2}$ of the submanifold F^n into the unit sphere S^{2n-2}. We call the map φ spherical and denote its image $T(\xi)$. The metric of $T(\xi)$ has the following form:

$$(d\xi)^2 = \sum_{i=1}^{n} \cos^2 \sigma_i \cos^2 \varphi_i (du^i)^2, \tag{4}$$

where φ_i is an angle between ξ and i-th principal vector of normal curvature $k_i, \quad i = 1, ..., n$. In the general case the spherical image cannot be regular and, moreover, it can degenerate in a submanifold of lower dimension than n.

We found a curvature tensor of the spherical image and proved a saddle character of spherical image, which considered as a submanifold in S^{2n-2}.

Theorem 4. *Under a spherical mapping the curvature lines are translated on the curvature lines of the spherical image, the asymptotic lines are translated on the asymptotic lines of submanifold $T(\xi) \subset S^{2n-2}$. The length of asymptotic lines is preserved under this mapping.*

IV. Section
Some new results with codimension > n-1.

In 1960 E.R Rozendorn in the work [11] constructs isometric immersion of complete Lobachevsky plane L^2 into E^5. His method is modification of the method of D.Blanusa, who gave the imbedding of L^2 into E^6(1954).

I investigate extrinsic-geometrical properties of the Rozendorn surface and proved the following theorem

Theorem 5. *The modul of the mean curvature vector H on the Rozendorn surface $L^2 \to E^5$ is bounded from above*

$$|H| \leq const.$$

In the work [12] D.V.Bolotov proved that does not exist a regular isometric immersion of L^n into Euclidean space E^{n+m} with flat normal connection and with $|H| \leq const.$

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NON-EUCLIDEAN GEOMETRY IN OBSERVER'S MATHEMATICS

D. Khots[1], **B. Khots**[2]

[1] 3710 S. 202nd Avenue Omaha NE, 68130, USA
[2] 4725 121st Street Des Moines, IA 50323, USA

Abstract

This work considers Geometrical and Physical aspects in a setting of arithmetic provided by Observer's Mathematics (see `www.mathrelativity.com`). We prove that Euclidean Geometry works in sufficiently small neighborhood of a given line, but when we enlarge the neighborhood, non-Euclidean Geometry takes over. We given an analog of the Lorentz Transform. We prove that the physical speed is a random variable, which cannot exceed some constant, and this constant does not depend on an inertial coordinate system. Certain results and communications pertaining to these theorems are also provided.

AMS Subject Classification: 51P05, 81T99

Key Words and Phrases: Observer, Geometry, arithmetic, derivative, Lorentz

I. Introduction

The following discussion is based on the work introduced in [1]. Further information can also be found in [2] and [3]. We consider a finite well-ordered system of observers, where each observer sees the real numbers as the set of all infinite decimal fractions. The observers are ordered by their level of "depth", i.e. each observer has a depth number (hence, we have the regular integer ordering), such that an observer with depth k sees that an observer with depth $n < k$ sees and deals (to be defined below) not with an infinite set of infinite decimal fractions, but, actually, with a finite set of finite decimal fractions. We call this set W_n, i.e. it is the set of all decimal fractions, such that there are at most than n digits in the integer part and n digits in the decimal part of the fraction. Visually, an element in W_n looks like $\underbrace{\_ \;\; ... \;\; \_}_{n} . \underbrace{\_ \;\; ... \;\; \_}_{n}$. Moreover, an observer with a given depth is unaware (or can only assume the existence) of observers with larger depth values and for his purposes, he deals with "infinity". These observers are called *naive*, with the observer with the lowest depth number – the most naive. However, if there is an observer with a higher depth number, he sees that a given observer actually deals with a finite set of finite decimal fractions, and so on. Therefore, if we fix an observer, then this observer sees the sets W_{n_1} ,. . . , W_{n_k} with $n_1 < ... < n_k$ indicating the depth level, and realizes that the corresponding observers see and deal with infinity. When we talk about observers, we shall always have some fixed observer (called 'us') who oversees all others and realizes that they are naive. The "W_n-observer" is the abbreviation for somebody who *deals* with W_n while thinking that he deals with infinity.

The following sections describe application of the idea of relativity in mathematics to various mathematical fields.

II. Arithmetic

We begin by defining sets W_n which consist of all finite decimal fractions such that there are at most n digits in the integer part and at most n digits in the decimal part. That is, the set W_n contains all elements of the form $a = a_0.a_1...a_n$ where the integer part can be written as $a_0 = b_{n-1}...b_0$, where $b_{n-1}, ..., b_0, a_1, ..., a_n \in \{0, 1, ..., 9\}$. If $n < m$, then W_n naturally embeds into W_m by placing 0's in the $n + 1^{st}$ through m^{th} decimal places. We call the embedding $\varphi_{n,m} : W_n \to W_m$. Here are some examples: let $2.34 \in W_2$ and then $\varphi_{2,4}(2.34) = 2.3400 \in W_4$. Similarly, W_m projects onto W_n by cutting off the superfluous digits on the right of the decimal point. Let $\varphi_{m,n} : W_m \to W_n$ be the projection, then, for example, if $45.4301 \in W_4$, then $\varphi_{4,2}(45.4301) = 45.43 \in W_2$. If the integer part of a fraction contains more than n digits, then $\varphi_{m,n}$ is not defined.

Now, given $c = c_0.c_1...c_n$, $d = d_0.d_1...d_n \in W_n$ we endow W_n with the following arithmetic $(+_n, -_n, \times_n, \div_n)$:

Definition 1. *Addition and subtraction*

$$c \pm_n d = \begin{cases} c \pm d, \text{ if } c \pm d \in W_n \\ \text{not defined, if } c \pm d \notin W_n \end{cases}$$

and we write $((...(c_1 +_n c_2)...) +_n c_N) = \sum\limits_{i=1}^{N} {}^n c_i$ *for* $c_1, ..., c_N$ *iff the contents of any parenthesis are in* W_n.

Definition 2. *Multiplication*

$$c \times_n d = \sum_{k=0}^{n} {}^n \sum_{m=0}^{n-k} {}^n 0.\underbrace{0...0}_{k-1} c_k \cdot 0.\underbrace{0...0}_{m-1} d_m$$

where $c, d \geq 0$, $c_0 \cdot d_0 \in W_n$, $0.\underbrace{0...0}_{k-1} c_k \cdot 0.\underbrace{0...0}_{m-1} d_m$ *is the standard product, and* $k = m = 0$ *means that* $0.\underbrace{0...0}_{k-1} c_k = c_0$ *and* $0.\underbrace{0...0}_{m-1} d_m = d_0$. *If either* $c < 0$ *or* $d < 0$, *then we compute* $|c| \times_n |d|$ *and define* $c \times_n d = \pm |c| \times_n |d|$, *where the sign* $\pm$ *is defined as usual. Note, if the content of at least one parentheses (in previous formula) is not in* W_n, *then* $c \times_n d$ *is not defined.*

Definition 3. *Division*

$$c \div_n d = \begin{cases} r, \text{ if } \exists!\ r \in W_n \quad r \times_n d = c \\ \text{not defined, if no such } r \text{ exists or it is not unique} \end{cases}$$

Let $n = 2$, so we are in W_2. Here are some examples of elements of W_2: $3.14, -99, 0.1 \in W_2$ and $0.115, 123.9, -100000 \notin W_2$. Now, the examples of arithmetic: $2.08 +_2 11.9 = 13.98$; $(-2.08) +_2 11.9 = 9.82$; $80 +_2 24 =$ not defined; $21.36 -_2 0.87 = 20.49$; $1.36 -_2 16.95 = -15.59$; $1.36 -_2 (-99.95) =$ not defined; $11 \times_2 8 = 88$; $(-5) \times_2 19 = -95$; $11 \times_2 12 =$ not defined; $3.41 \times_2 2.64 = 8.98$; $3.41 \times_2 (-2.64) = -8.98$; $3.41 \times_2 42.64 =$ not defined; $99.41 \times_2 1.64 =$ not defined; $0.85 \times_2 0.02 = 0$; $80 \div_2 4 = 20$; $1 \div_n 0.5 =$ not defined (since we get 10 different r's); $1 \div_n 3 =$ not defined (since no r exists).

III. Derivatives

From the point of view of W_n-observer (we will call such observers "naive", since they "think" that they "live" in W and deal with W) a real function y of a real variable x, $y = y(x)$, is called differentiable at $x = x_0$ (see [4]) if there is a derivative

$$y'(x_0) = \lim_{x \to x_0, x \neq x_0} \frac{y(x) - y(x_0)}{x - x_0}$$

What does the above statement mean from point of view of W_m-observer with $m > n$? It means that

$$|(y(x) -_n y(x_0)) -_n (y'(x_0) \times_n (x -_n x_0))| \leq 0.\underbrace{0\ldots01}_{n}$$

whenever

$$|y(x) -_n y(x_0)| = 0.\underbrace{0\ldots0y_l}_{l}\, y_{l+1}\ldots y_n$$

and

$$|(x -_n x_0)| = 0.\underbrace{0\ldots0x_k}_{k}\, x_{k+1}\ldots x_n$$

for $1 \leq k$, $l \leq n$, and x_k - non-zero digit.

We now state the main theorems.

Theorem 1. *From the point of view of a W_m-observer a derivative calculated by a W_n-observer ($m > n$) is not defined uniquely.*

Proof. Put $y'(x_0) = \pm a_0.a_1\ldots a_p a_{p+1}\ldots a_n$ with $a_0.a_1\ldots a_p a_{p+1}\ldots a_n \geq 0$ and $p \leq n$. Then $0.\underbrace{0\ldots0y_l}_{l}\, y_{l+1}\ \ldots y_n = a_0.a_1\ \ldots a_p a_{p+1}\ \ldots a_n \times_n 0.\underbrace{0\ldots0x_k}_{k}\, x_{k+1}\ldots x_n = a_0.a_1\ldots a_p b_{p+1}\ldots b_n \times_n 0.\underbrace{0\ldots0x_k}_{k}\, x_{k+1}\ldots x_n$ for any digits $b_{p+1},\ldots,b_n$ and $p = n - k$. Hence $y'(x_0) \in V = \{\pm a_0.a_1\ldots a_p\ a_{p+1}\ \ldots\ a_n | a_{p+1},\ \ldots, a_n \in \{0,1,\ldots,9\}\}$ and $|V| = 10^k$. QED.

Theorem 2. *From the point of view of a W_m-observer with $m > n$, $|y'(x_0)| \leq C_n^{l,k}$, where $C_n^{l,k} \in W_n$ is a constant defined only by n, l, k and not dependent on $y(x)$.*

Proof. We have $\pm 0.\underbrace{0\ldots 0y_l}_{l} y_{l+1}\ldots y_n = (\pm a_0.a_1\ldots a_n)\times_n (\pm 0.\underbrace{0\ldots 0x_k}_{k}$ $x_{k+1}\ldots x_n)$ with x_k - non-zero digit and $a_0.a_1\ldots a_p a_{p+1}\ldots a_n \geq 0$. Now, if $l > k$ then $a_0 = 0$; if $l = k$ then $a_0 \leq 9$ and if $l < k$ then $a_0 < 9\times 10^{k-1}$. Hence

$$C_n^{l,k} = \begin{cases} 1, \text{ if } l > k \\ 10, \text{ if } l = k \\ 9\times 10^{k-1}, \text{ if } l < k \end{cases}$$

QED.

Theorem 3. *From the point of view of a W_m-observer, when a W_n-observer (with $m > n \geq 3$) calculates the second derivative:*

$$y''(x_0) = \lim_{x_1\to x_0, x_1\neq x_0, x_2\to x_0, x_2\neq x_0, x_3\to x_1, x_3\neq x_1} \frac{\frac{y(x_3)-y(x_1)}{(x_3-x_1)} - \frac{y(x_2)-y(x_0)}{x_2-x_0}}{x_1 - x_0}$$

we get the following unequality:

$$(|x_2 -_n x_0| \times_n |x_3 -_n x_1|) \times_n |x_1 -_n x_0| \geq 0.\underbrace{0\ldots 01}_{n}$$

provided that $y''(x_0) \neq 0$.

Proof. For the W_m-observer existence of $y''(x_0)$ means that $|((y(x_3) -_n y(x_1))\times_n (x_2 -_n x_0) -_n ((y(x_2) -_n y(x_0))\times_n (x_2 -_n x_0))) -_n y''(x_0)\times_n ((|x_2 -_n x_0| \times_n |x_3 -_n x_1|) \times_n |x_1 -_n x_0|)| \leq 0.\underbrace{0\ldots 01}_{n}$, whenever

$$|(x_2 -_n x_0)| \leq 0.\overbrace{\underbrace{0\ldots 0p}_{k} *\ldots *}^{n}$$

and

$$|(x_3 -_n x_1)| \leq 0.\overbrace{\underbrace{0\ldots 0q}_{l} *\ldots *}^{n}$$

and

$$|(x_1 -_n x_0)| \leq 0.\overbrace{\underbrace{0\ldots0}_{s}r*\ldots*}^{n}$$

where p, q, r are non-zero digits, asterisks are any digits and $3 \leq k+l+s \leq n$. Then given $y''(x_0) \neq 0$ we have $(|x_2 -_n x_0| \times_n |x_3 -_n x_1|) \times_n |x_1 -_n x_0| \geq 0.\underbrace{0\ldots01}_{n}$. QED.

IV. Physical Interpretation

The following hypotheses illustrate possible physical interpretation of previous theorems.

Hypotheses 1 Theorem 1 could offer an explanation of why physical speed (or acceleration) is not uniquely defined and, from the point of view of a measurement system (observer), it is possible to consider speed (or acceleration) as a random variable with distribution dependend on the measurement system. Let v be the speed with $v = v_0.v_1 \ldots v_{n-k} + \xi_m^{n,k}$ where $\xi_m^{n,k} \in \{0.\underbrace{0\ldots0}_{n-k}v_{n-k+1}\ldots v_n\}$ - random variable, $m > n$, and the distribution function is $F_m^{n,k}(x) = P(\xi_m^{n,k} < x)$.

Hypotheses 2 Theorem 2 could offer an explanation of why the speed of any physical body cannot exceed some constant, (the speed of light, for example). Independence of this constant on explicit expression of space-time function could offer an explanation of why the speed of light does not depend on an inertial coordinate system.

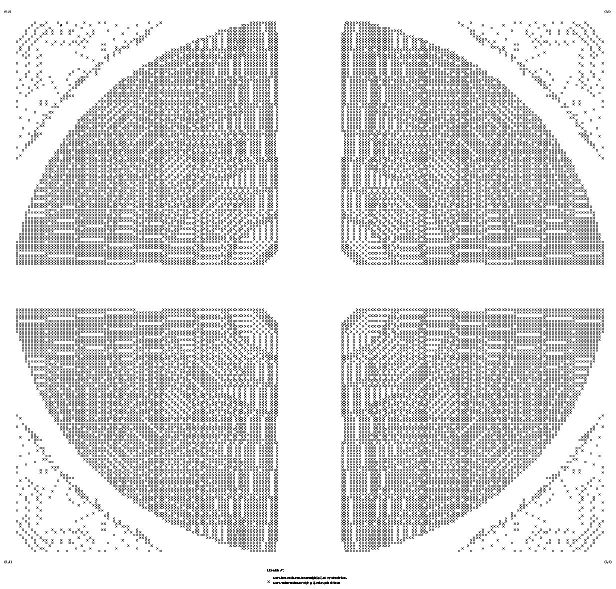
Hypotheses 3 Theorem 3 could offer an explantion of the various uncertainty principles, when a product of a finite number of physical variables has to be not less than a certain constant. This can be seen not just from consideration of second derivatives, but of any derivative.

Hypotheses 4 Theorems 1, 2, and 3 combined may provide an insight into the connection between classical and quantum mechanics.

V. Nadezhda Effect

In this section we consider an open square Q centered at the origin with sides of length 2 located on a plane $W_n \times W_n$. We will calculate the distance D between the origin $(0,0)$ and any point of Q as follows. $D = \rho((0,0),(x,y)) = \sqrt{x^2+y^2} = \sqrt{x \times_n x +_n y \times_n y}$, where $\sqrt{a} = b$ means $b \times_n b = a$, $x, y \in Q$, i.e., $|x| < 1$, $|y| < 1$.

The figure below contains an illustration of the fact that for some points on $W_n \times W_n$ the concept of distance from the origin does not exist; while for others it does exist. The illustration below is for $n = 3$ ($Q \subset W_3 \times W_3$). Points with no distance to the origin are indicated by black, while points where distance from the origin exists are indicated in white.



This means that the distance D does not always exist, i.e., not every segment on a plane has a length. This phenomenon occurs for all n. We call the presence of these "black holes" as the Nadezhda Effect. This effect gives us new possibilities for discovering physical processes and developing their mathematical models.

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POLYHEDRAL SPACE FORMS WITH HYPERBOLIC AND OTHER METRICS

E. Molnár

Budapest University of Technology
Economics Institute of Mathematics, Department of Geometry

Abstract

In earlier works of the author, partly joint with I. Prok and J. Szirmai (e.g. [M92], [M97], [M05], [MPSz06]), the projective sphere $PS^d(\mathbf{R}; \mathbf{V}^{d+1}; \boldsymbol{V}_{d+1}; +)$ has been introduced for presentation of polyhedral d-orbifolds and d-manifolds, mainly in the homogeneous 3-spaces

$$\mathbf{E}^3;\ \mathbf{S}^3;\ \mathbf{H}^3;\ \mathbf{S}^2\!\times\!\mathbf{R};\ \mathbf{H}^2\!\times\!\mathbf{R};\ \widetilde{\mathbf{SL_2R}};\ \mathbf{Nil};\ \mathbf{Sol}$$

(Thurston geometries). The main steps can be indicated as follows.

1. A projective simplex coordinate system has to be introduced for the fundamental polyhedron, where the face pairing generators are expressed by linear mappings upto projective freedom with some free parameters.
2. The defining relations for the symmetry groups (by the induced edge equivalence classes) fix some parameters of the generator mappings, by matrix equations, occasionally of high degree.
3. We look for a plane-point polarity (or scalar product) for the orthogonality of planes of a 3-dimensional projective metric geometry from the eight possibilities above. This polarity (i.e. the orthogonality of planes) has to be invariant under the generator mappings. These lead to linear matrix equations for the symmetric polarity matrix.

4. The signature of polarity (scalar product, fundamental quadratic form), if it is not trivial, with some additional properties, provides the possible Thurston geometry.

5. If the signature is $(0; +; +; +)$, then we obtain Euclidean 3-tiling with exact matrices for the generators and the scalar product, possibly with free parameters. Moreover, by a conventional coordinate system we can recognize the corresponding crystallographic space group as well.

6. Other signatures (e.g. $(+; +; +; +)$ to spherical space, $(-; +; +; +)$ to hyperbolic or Bolyai-Lobachevskii space) lead to other realizations. Or - if only trivial polarity is possible - then either certain "splitting effects" occur, or the famous Thurston conjecture would not be true (!), considered still to be open, in general (?).

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ON NON-EQUIVALENT FUNCTIONAL BASES OF FIRST-ORDER DIFFERENTIAL INVARIANTS OF THE NON-CONJUGATE SUBGROUPS OF THE POINCARÉ GROUP $P(1,4)$

V. M. Fedorchuk[1], **V. I. Fedorchuk**[2]

[1] Pedagogical Academy, Institute of Mathematics, 2 Podchorążych Street, 30-084 Kraków, Poland; Pidstryhach Institute of Applied Problems of Mechanics and Mathematics, National Ukrainian Academy of Sciences, 3b Naukova Street, 79-053 L'viv, Ukraine

[2] Pidstryhach Institute of Applied Problems of Mechanics and Mathematics, National Ukrainian Academy of Sciences, 3b Naukova Street, 79-053 L'viv, Ukraine

Abstract

The functional bases of the first-order differential invariants of all non-conjugate subgroups of the Poincaré group $P(1,4)$ have been divided into classes of equivalent bases. The number of all non-equivalent functional bases has been determined. The application of the results obtained to the construction of classes of the first-order differential equations in the space $M(1,3) \times R(u)$ invariant under these subgroups is discussed. Among those classes, there are some invariant under the following subgroups of the group $P(1,4)$: $SO(2)$, $SO(3)$, $E(2)$, $E(3)$, $SO(1,3)$, $SO(4)$, $E(4)$, $P(1,3)$, $SO(1,4)$, $\widetilde{G}(1,3)$, etc.

I. Introduction

In many cases, mathematical models of various processes can be described by means of differential equations (linear or nonlinear) in the spaces of different dimensions and different types (Euclidean, non-Euclidean, etc.).

It is well known (see, for example, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]) that the majority of differential equations, which are useful in theoretical and mathematical physics, mechanics, gas dynamics have non-trivial symmetry groups. For example, in the space $M(1,3) \times R(u)$, we have the following equations:

1. The Eikonal equation

$$u^{\mu}u_{\mu} \equiv (u_0)^2 - (u_1)^2 - (u_2)^2 - (u_3)^2 = 1,$$

where $u = u(x)$, $x = (x_0, x_1, x_2, x_3) \in M(1,3)$, $u_{\mu} \equiv \dfrac{\partial u}{\partial x_{\mu}}$, $u^{\mu} = g^{\mu\nu}u_{\nu}$, $\mu, \nu = 0, 1, 2, 3.$

2. The Euler-Lagrange-Born-Infeld equation

$$\Box u\,(1 - u_{\nu}u^{\nu}) + u^{\mu}u^{\nu}u_{\mu\nu} = 0,$$

where $u = u(x), x = (x_0, x_1, x_2, x_3) \in M(1,3)$, $u_{\mu} \equiv \dfrac{\partial u}{\partial x^{\mu}}$, $u_{\mu\nu} \equiv \dfrac{\partial^2 u}{\partial x^{\mu}\partial x^{\nu}}$, $u^{\mu} = g^{\mu\nu}u_{\nu}$, $g_{\mu\nu} = (1, -1, -1, -1)\delta_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$, $\Box$ is the d'Alembert operator.

3. The homogeneous Monge-Ampère equation

$$\det\,(u_{\mu\nu}) = 0,$$

where $u = u(x)$, $x = (x_0, x_1, x_2, x_3) \in M(1,3)$, $u_{\mu\nu} \equiv \dfrac{\partial^2 u}{\partial x_{\mu}\partial x_{\nu}}$, $\mu, \nu = 0, 1, 2, 3.$

4. The inhomogeneous Monge-Ampère equation

$$\det\,(u_{\mu\nu}) = \lambda\,(1 - u_{\nu}u^{\nu})^3, \quad \lambda \neq 0,$$

where $u = u(x)$, $x = (x_0, x_1, x_2, x_3) \in M(1,3)$, $u_{\mu\nu} \equiv \dfrac{\partial^2 u}{\partial x_\mu \partial x_\nu}$,
$u^\nu = g^{\nu\alpha} u_\alpha$, $u_\alpha \equiv \dfrac{\partial u}{\partial x_\alpha}$, $g_{\mu\nu} = (1, -1, -1, -1)\delta_{\mu\nu}$, $\mu, \nu, \alpha = 0, 1, 2, 3.$

Here, and in what follows, $M(1,3)$ is a four-dimensional Minkowski space; $R(u)$ is a real number axis of the depended variable u.

These equations are invariant under generalized Poincaré group $P(1,4)$ (see, for example, [7, 12, 13]). The group $P(1,4)$ is a group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. This group has many applications in theoretical and mathematical physics (see, for example, [9, 14]). Continuous subgroups of the group $P(1,4)$ have been found in [15, 16, 17]. One of the nontrivial consequences of the description of the non-conjugate subalgebras of the Lie algebra of the group $P(1,4)$ is that the Lie algebra of the group $P(1,4)$ contains, as subalgebras, the Lie algebra of the Poincaré group $P(1,3)$ and the Lie algebra of the extended Galilei group $\widetilde{G}(1,3)$ [9, 18], i.e. it naturally unites the Lie algebras of the symmetry groups of relativistic and non-relativistic physics. Therefore, the construction of the classes of differential equations, which are defined in the space $M(1,3) \times R(u)$ and invariant under non-conjugate subgroups of the group $P(1,4)$, is important from different points of view.

In many cases (see, for example, [3, 5, 19]), these classes can be written in the following form:

$$F(J_1, J_2, ..., J_t) = 0, \tag{1.1}$$

where F is an arbitrary smooth function of its arguments, $\{J_1, J_2, ..., J_t\}$ is functional basis of differential invariants of the corresponding subgroup of the group $P(1,4)$.

It should be noted that each of these classes is a non-singular differential invariant manifold of the corresponding non-conjugate subalgebra of the Lie algebra of the group $P(1,4)$. More details on the manifolds of this type can be found in [3, 5].

As we see from the formula (1.1), the properties of these classes essentially depend on the properties of the corresponding functional bases.

The construction of functional bases of differential invariants for non-conjugate subgroups of different Lie groups has shown that there is no one-to-one correspondence between the non-conjugate subgroups of these groups and the corresponding to them functional bases of differential invariants. It means that the different non-conjugate subgroups of Lie groups can have the same (equivalent) functional bases of differential invariants.

In [20, 21] we have presented some results, which referred to the application of equivalence criterion [20, 22] in order to construct separately all non-equivalent functional bases of the first-order differential invariants of splitting and non-splitting subgroups of the group $P(1,4)$.

The purpose of the present paper is to give some new results obtained by means of equivalence criterion for functional bases of the first-order differential invariants of all non-conjugate subgroups of the group $P(1,4)$.

II. The Lie algebra of the group $P(1,4)$ and its representation.

The Lie algebra of the group $P(1,4)$ is given by the 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}$ $(\mu, \nu = 0,1,2,3,4)$ and P'_μ $(\mu = 0,1,2,3,4)$, satisfying the commutation relations

$$\left[P'_\mu, P'_\nu\right] = 0, \qquad \left[M'_{\mu\nu}, P'_\sigma\right] = g_{\mu\sigma}P'_\nu - g_{\nu\sigma}P'_\mu,$$

$$\left[M'_{\mu\nu}, M'_{\rho\sigma}\right] = g_{\mu\rho}M'_{\nu\sigma} + g_{\nu\sigma}M'_{\mu\rho} - g_{\nu\rho}M'_{\mu\sigma} - g_{\mu\sigma}M'_{\nu\rho},$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$. Here, and in what follows, $M'_{\mu\nu} = iM_{\mu\nu}$.

In the following we will use new basis elements

$$G = M'_{40}, \quad L_1 = M'_{32}, \quad L_2 = -M'_{31}, \quad L_3 = M'_{21},$$

$$P_a = M'_{4a} - M'_{a0}, \quad C_a = M'_{4a} + M'_{a0}, \quad (a = 1,2,3),$$

$$X_0 = \frac{1}{2}\left(P'_0 - P'_4\right), \quad X_k = P'_k \quad (k = 1,2,3), \quad X_4 = \frac{1}{2}\left(P'_0 + P'_4\right).$$

All non-conjugate subalgebras of the Lie algebra of the group $P(1,4)$ are divided into splitting and non-splitting ones. More details on the splitting and non-splitting subalgebras of any finite-dimensional Lie algebra can be found in [23].

Splitting subalgebras $P_{i,a}$ of the Lie algebra of the group $P(1,4)$ can be written in the following form:

$$P_{i,a} = F_i \overset{\circ}{+} N_{ia},$$

where F_i are subalgebras of the Lie algebra of the group $O(1,4)$, N_{ia} are subalgebras of the Lie algebra of the translation group $T(5) \in P(1,4)$.

Non-splitting subalgebras $\widetilde{P}_{j,k}$ are subalgebras, for which basis can be chosen in the form:

$$\widetilde{B}_k = B_k + \sum_i c_{ki} X_i, \qquad \sum_j d_{rj} X_j,$$

where c_{ki} and d_{rj} are fixed real constants (not equal zero simultaneously). B_k are bases of subalgebras of the Lie algebra of the group $O(1,4)$, X_i are bases of subalgebras of the Lie algebra of the group $T(5)$.

Let us consider the following representation of the Lie algebra of the group $P(1,4)$:

$$P'_0 = \frac{\partial}{\partial x_0}, \quad P'_1 = -\frac{\partial}{\partial x_1}, \quad P'_2 = -\frac{\partial}{\partial x_2}, \quad P'_3 = -\frac{\partial}{\partial x_3},$$

$$P'_4 = -\frac{\partial}{\partial u}, \quad M'_{\mu\nu} = -\left(x_\mu P'_\nu - x_\nu P'_\mu\right), \ x_4 \equiv u.$$

It means that the group $P(1,4)$ acts on the space $M(1,3) \times R(u)$. More details about the representations of this type can be found in [7, 12, 13].

III. On non-equivalent functional bases of the first-order differential invariants of non-conjugate subgroups of the group $P(1,4)$.

In this section we consider the construction of non-equivalent functional bases of the first-order differential invariants of non-conjugate subgroups of the group $P(1,4)$, as well as the application of them in order to construct mathematical models (differential equations) with nontrivial symmetry groups in the space $M(1,3)\times R(u)$.

Let $\{J_1^{(1)}, J_2^{(1)}, ..., J_t^{(1)}\}$ and $\{J_1^{(2)}, J_2^{(2)}, ..., J_t^{(2)}\}$ be functional bases of the first-order differential invariants, which correspond to the non-conjugate subalgebras L^1 and L^2 of the Lie algebra of the group $P(1,4)$.

Definition. We say that the functional bases $\{J_1^{(1)}, J_2^{(1)}, ..., J_t^{(1)}\}$ and $\{J_1^{(2)}, J_2^{(2)}, ..., J_t^{(2)}\}$ be *equivalent* if there exist smooth functions $f_1, f_2, ..., f_t$ and $g_1, g_2, ..., g_t$ such that

$$\begin{array}{lcl}
J_1^{(2)} = f_1(J_1^{(1)}, J_2^{(1)}, ..., J_t^{(1)}) & & J_1^{(1)} = g_1(J_1^{(2)}, J_2^{(2)}, ..., J_t^{(2)}) \\
J_2^{(2)} = f_2(J_1^{(1)}, J_2^{(1)}, ..., J_t^{(1)}) & & J_2^{(1)} = g_2(J_1^{(2)}, J_2^{(2)}, ..., J_t^{(2)}) \\
\dots\dots\dots\dots\dots\dots & \text{and} & \dots\dots\dots\dots\dots\dots \\
J_t^{(2)} = f_t(J_1^{(1)}, J_2^{(1)}, ..., J_t^{(1)}) & & J_t^{(1)} = g_t(J_1^{(2)}, J_2^{(2)}, ..., J_t^{(2)}).
\end{array}$$

Proposition 1. Two functional bases $\{J_1^{(1)}, J_2^{(1)}, ..., J_t^{(1)}\}$ and $\{J_1^{(2)}, J_2^{(2)}, ..., J_t^{(2)}\}$ are equivalent if and only if they satisfy the following conditions:

$$\begin{array}{l}
\widetilde{X}_1^{(1)} J_1^{(2)} = 0, \widetilde{X}_1^{(1)} J_2^{(2)} = 0, ..., \widetilde{X}_{r_1}^{(1)} J_t^{(2)} = 0 \\
\widetilde{X}_1^{(2)} J_1^{(1)} = 0, \widetilde{X}_1^{(2)} J_2^{(1)} = 0, ..., \widetilde{X}_{r_2}^{(2)} J_t^{(1)} = 0
\end{array} \tag{3.1}$$

where $\{\widetilde{X}_1^{(1)}, \widetilde{X}_2^{(1)}, ..., \widetilde{X}_{r_1}^{(1)}\}$, $\{\widetilde{X}_1^{(2)}, \widetilde{X}_2^{(2)}, ..., \widetilde{X}_{r_2}^{(2)}\}$ are the first-prolonged bases operators of the Lie subalgebra L^1 and L^2, respectively; r_1, r_2 are the dimensions of the subalgebras L^1 and L^2.

Proof. The Proof of this Proposition for splitting subalgebras of the Lie algebra of the group $P(1,4)$ can be found in [20] (see Lemma). Since the proof of this Proposition for non-conjugate subalgebras of the Lie algebra of the group $P(1,4)$ is quite analogical to the case of splitting subalgebras, therefore we omit it here. The generalization of this Proposition on the functional bases of any finite order differential invariants of the non-conjugate subgroups of local Lie groups of the point transformations can be found in [22].

We have used this Proposition as the criterion of the equivalence for any two functional bases of the first-order differential invariants of the non-conjugate subgroups of the group $P(1,4)$.

Proposition 2. *There exist 494 non-equivalent functional bases of the first-order differential invariants for the non-conjugate subgroups of the group* $P(1,4)$.

Sketch of proof. The list of all non-conjugate (the conjugation was considered under the group $P(1,4)$) subalgebras of the Lie algebra of the group $P(1,4)$ contains 555 ones [4].

As following from the calculation of the general ranks of matrices, which contain coordinates of the one-prolonged basis elements of the subalgebras of the Lie algebra considered, and using the theorem on number of invariants of the Lie group of the point transformations (see, for example, [5, 3]) we make sure that the 550 of the non-conjugate subalgebras of the Lie algebra of the group $P(1,4)$ have the functional bases of the first-order differential invariants. Thus, there are 550 functional bases of the first-order differential invariants. Among them, there are equivalent ones. Equivalent functional bases can only be among those, which have the same dimensions.

Let L^1 be a non-conjugate subalgebra of the Lie algebra of the group $P(1,4)$, which has the t-dimensional functional basis of the first-order differential invariants $\{J_1^{(1)}, J_2^{(1)}, ..., J_t^{(1)}\}$. To find the bases, which are equivalent to $\{J_1^{(1)}, J_2^{(1)}, ..., J_t^{(1)}\}$, we use the Proposition 1. Let $\{J_1^{(2)}, J_2^{(2)}, ..., J_t^{(2)}\}$ be t-dimensional functional basis of the first-order differential invariants of the other non-conjugate subalgebra L^2. Following the Proposition 1, if these functional bases satisfy the conditions (3.1), then, the considered bases are equivalent. Otherwise, the considered bases are not equivalent. In the analogous manner, we check whether other t-dimensional functional bases of the first-order differential invariants are equivalent to the $\{J_1^{(1)}, J_2^{(1)}, ..., J_t^{(1)}\}$ or not. In this way, we obtain all t-dimensional functional bases, which are equivalent to $\{J_1^{(1)}, J_2^{(1)}, ..., J_t^{(1)}\}$.

In the analogous manner, we construct classes of the equivalent functional bases of other dimensions.

The direct application of the mentioned above criterion give us 494 non-equivalent functional bases of the first-order differential invariants for the non-conjugate subgroups of the group $P(1,4)$. The Proposition is proved.

Taking into account the non-equivalent functional bases of the first-order differential invariants of the non-conjugate subgroups of the group $P(1,4)$ we can construct 494 classes of the first-order differential equations, which are defined in the space $M(1,3) \times R(u)$ and invariant under the non-conjugate subgroups of this group. All these classes of equations can be written in the form (1.1).

It is impossible to present all these classes here. Below, only for the Lie algebras of some subgroups of the group $P(1,4)$, often applicable in theoretical and mathematical physics, we write their basis elements and corresponding classes of the first-order differential equations in the space $M(1,3) \times R(u)$.

1. $\langle L_3 \rangle (\cong SO(2))$,

$$F\left(x_0,\ x_3,\ (x_1^2+x_2^2)^{1/2},\ u,\ x_1u_2 - x_2u_1,\ u_0,\ u_3,\ u_1^2+u_2^2\right) = 0,$$

$$u_\mu \equiv \frac{\partial u}{\partial x_\mu}, \quad \mu = 0,1,2,3;$$

2. $\langle L_1,\ L_2,\ L_3 \rangle (\cong SO(3))$,

$$F\left(x_0,\ (x_1^2+x_2^2+x_3^2)^{1/2}, u, u_0, x_1u_1 + x_2u_2 + x_3u_3,\ u_1^2+u_2^2+u_3^2\right) = 0\,;$$

3. $\langle L_3,\ X_1,\ X_2 \rangle (\cong E(2))$,

$$F\left(x_0,\ x_3,\ u,\ u_0,\ u_3,\ u_1^2+u_2^2\right) = 0\ ;$$

4. $\langle L_1,\ L_2,\ L_3,\ X_1,\ X_2,\ X_3\rangle(\cong E(3))$,

$$F\left(x_0,\ u,\ u_0,\ u_1^2+u_2^2+u_3^2\right)=0\ ;$$

5. $\langle L_1,\ L_2,\ L_3,\ P_1-C_1,\ P_2-C_2,\ P_3-C_3\rangle(\cong SO(1,3))$,

$$F\left((x_0^2-x_1^2-x_2^2-x_3^2)^{1/2},\ u,\ x_0u_0+x_1u_1+x_2u_2+x_3u_3,\right.$$
$$\left.u_0^2-u_1^2-u_2^2-u_3^2\right)=0\ ;$$

6. $\langle L_1,\ L_2,\ L_3,\ P_1+C_1,\ P_2+C_2,\ P_3+C_3\rangle(\cong SO(4))$,

$$F\left(x_0,\ (x_1^2+x_2^2+x_3^2+u^2)^{1/2},\ \frac{x_1u_1+x_2u_2+x_3u_3-u}{u_0},\right.$$
$$\left.\frac{u_1^2+u_2^2+u_3^2+1}{u_0^2}\right)=0\ ;$$

7. $\langle L_1,\ L_2,\ L_3,\ P_1{+}C_1,\ P_2{+}C_2,\ P_3{+}C_3,\ X_1,\ X_2,\ X_3,\ X_0{-}X_4\rangle(\cong E(4))$,

$\langle L_1+\frac{1}{2}\left(P_1+C_1\right),\ L_2+\frac{1}{2}\left(P_2+C_2\right),\ L_3+\frac{1}{2}\left(P_3+C_3\right),\ X_1,\ X_2,$
$X_3,\ X_0-X_4\rangle$,

$\langle L_1{+}\frac{1}{2}\left(P_1+C_1\right),\ L_2{+}\frac{1}{2}\left(P_2+C_2\right),\ L_3{+}\frac{1}{2}\left(P_3+C_3\right),\ L_3{-}\frac{1}{2}\left(P_3+C_3\right),$
$X_1,\ X_2,\ X_3,\ X_0-X_4\rangle$,

$$F\left(x_0,\ \frac{u_1^2+u_2^2+u_3^2+1}{u_0^2}\right)=0\ ;$$

8. $\langle L_1,\ L_2,\ L_3,\ P_1{-}C_1,\ P_2{-}C_2,\ P_3{-}C_3,\ X_1, X_2,\ X_3,\ X_0{+}X_4\rangle(\cong P(1,3))$,

$$F\left(u,\ u_0^2-u_1^2-u_2^2-u_3^2\right)=0\ ;$$

9. $\langle G,\ C_1,\ C_2,\ C_3,\ L_1,\ L_2,\ L_3,\ P_1,\ P_2,\ P_3\rangle(\cong SO(1,4))$,

$$F\left((x_0^2-x_1^2-x_2^2-x_3^2-u^2)^{1/2}, \frac{(x_0u_0+x_1u_1+x_2u_2+x_3u_3-u)^2}{u_0^2-u_1^2-u_2^2-u_3^2-1}\right)=0\,;$$

10. $\langle L_1,\ L_2,\ L_3,\ P_1,\ P_2,\ P_3,\ X_0,\ X_1,\ X_2,\ X_3,\ X_4\rangle(\cong \widetilde{G}(1,3))$,

$\langle P_1,\ P_2,\ P_3,\ X_0,\ X_1,\ X_2,\ X_3,\ X_4\rangle$,

$\langle L_3-P_3,\ P_1,\ P_2,\ X_0,\ X_1,\ X_2,\ X_3,\ X_4\rangle$,

$$\langle L_3,\ P_1,\ P_2,\ P_3,\ X_0,\ X_1,\ X_2,\ X_3,\ X_4\rangle,$$
$$\langle L_3 - X_0,\ P_1,\ P_2,\ P_3,\ X_1,\ X_2,\ X_3,\ X_4\rangle,$$
$$\langle P_1,\ P_2,\ P_3 + X_0,\ L_3 + \beta X_0,\ X_1,\ X_2,\ X_3,\ X_4,\ \beta < 0\rangle,$$
$$F\left(\frac{u_1^2 + u_2^2 + u_3^2 + 2(u_0+1)}{(u_0+1)^2}\right) = 0\ .$$

Since the Lie algebra of the group $P(1,4)$ contains, as subalgebras, the Lie algebra of the Poincaré group $P(1,3)$ and the Lie algebra of the extended Galilei group $\widetilde{G}(1,3)$ (see also [9, 18]), the obtained classes of differential equations can be used in relativistic and non-relativistic physics.

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ITÔ-STRATONOVITCH FORMULA FOR A FOUR ORDER OPERATOR ON A TORUS

R. Léandre

Institut de Mathématiques. Université de Bourgogne. 21000. Dijon. FRANCE

Abstract

We give an Itô-Stratonovitch formula for a semi-group generated by a four order operator on a torus.

I. Introduction

Let B_t a Brownian motion on $\mathbb{R}$. By the celebrated Itô formula ([2]), we have if f is a C^2 function from $\mathbb{R}$ into $\mathbb{R}$:

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)\delta B_s + 1/2\int_0^t f"(B_s)ds \tag{1}$$

where δB_s is the Itô differential.

This formula can be convert in the Stratonovitch Calculus in

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s \tag{2}$$

where dB_s is the Stratonovitch differential.

Itô-Stratonovitch formula for diffusion processes was translated in semi-group theory by Léandre ([11]). Léandre ([3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15]) has translated in semi-group theory a lot of tools of stochastic analysis, by using the classical relation between the theory of stochastic processes and the theory of Markovian semi-groups, such that the tools of stochastic analysis become algebraic comptations on the semi-group, the estimates being done because we get semi-groups in probability measures.

It is interesting to developp this strategy when we consider more general semi group: it is the purpose of this communication to do that in a simple case.

II. Statement of the main theorem

We consider a torus $\mathbb{T}^n$ ($x \in \mathbb{T}^n$) and a orthonormal basis of its Lie algebra ∂_i. We consider the four order elliptic oeparator $\Delta = \sum(\partial_i)^4$. It is symmetric positive self-adjoint. It generates a (non-markovian!) semi-group P_t on $L^2(\mathbb{T}^n)$, the torus being endowed of its Haar measure.

We consider a smooth function f from $\mathbb{T}^n$ into $\mathbb{R}$ and the vector field on $\mathbb{T}^n \times \mathbb{R}$, $(x, y) \in \mathbb{T}^n \times \mathbb{R}$

$$\partial_i^f = \partial_i + (\partial_i f)\partial_y \tag{3}$$

and we consider the degenerated operator on $\mathbb{T}^n \times \mathbb{R}$

$$\Delta^f = \sum(\partial_i^f)^4 \tag{4}$$

It is symmetric positive, and therefore has a self-adjoint extension on $L^2(\mathbb{T}^n \times \mathbb{R})$, $\mathbb{T}^n \times \mathbb{R}$ being endowed of its Haar measure. This self-adjoint extension Δ^f generates therefore a semi-group P_t^f on $L^2(\mathbb{T}^n \times \mathbb{R})$.

We consider a smooth function $g(.,.)$ from $\mathbb{T}^n \times \mathbb{R}$ with compact support and the function on $\mathbb{T}^n$

$$g^f(x) = g(x, f(x)) \tag{5}$$

Our main theorem is:

Theorem*(Itô-Stratonovitch) We have the relation*

$$P_t[g^f](x) = P_t^f[g(.,.)](x, f(x)) \tag{6}$$

III. Proof of the theorem

It follows the same strategy of the proof of the Itô-Stratonovitch formula of [11], the difficulty being that for the estimates we consider a Non-Markovian semi-group, the algebra being more at less the same.

We suppose first of all that f is a finite sum of trigonometric and that g is a finit sum of a product of trigometric function in x and expression of the type $y^n \exp[-ay^2]$ $a > 0$. In such a case,

$$P_t[g^f](x) = g^f(x) + \sum \frac{t^n}{n!}(-\Delta)^n g^f(x) \tag{7}$$

But if we consider an expression ψ which depends only on x, we have

$$\partial_i(g^f \psi) = (\partial_i^f(g(.,.)\psi)(x, f(x)) \tag{8}$$

such that we recognize in the right hand side of (7)

$$g(x, f(x)) + \sum \frac{t^n}{n!}(-\Delta^f)^n(g(.,.))(x, f(x)) = P_t^f[g(.,.)](x, f(x)) \tag{9}$$

Since the continuous semi-groups are continuous in L^2, the formula (6) is valid for all smooth $g(.,.)$ with compact supports.

The theorem comes then from the following lemma:

Lemma*If f_n as well as all its derivatives tend to f uniformly, and if g is smooth with compact support, then*

$$P_t^{f_n}[g(.,.)](x, y) \to P_t^f[g(.,.)](x, y) \tag{10}$$

uniformly and in $L^2(\mathbb{T}^n \times \mathbb{R})$.

Proof: We remark that the vector fields ∂_i^f commute and that $\partial_y = \partial_0^f$ commutes with them. Moreover if the supremum norm of the k^{th} derivatives of f are bounded, $\partial_i^f, i = 0, .., n$ constitute uniformly a basis of the tangent space of $\mathbb{T}^n \times \mathbb{R}$. Let $(\alpha) = (\alpha_0, .., \alpha_n)$ be a multiindex and $(\partial^f)^{(\alpha)}$ be the associated differential operator. It is enough to show that

$$(\partial^{f_n})^{(\alpha)} P_t^{f_n}[g(.,.)](x, y) \to (\partial^f)^{(\alpha)} P_t^f[g(.,.)](x, y) \tag{11}$$

uniformly and in L^2. But $(\partial^{f_n})^{(\alpha)}$ commute with Δ^{f_n} such that

$$(\partial^{f_n})^{(\alpha)} P_t^{f_n}[g(.,.)](x,y) = P_t^{f_n}[(\partial^{f_n})^{(\alpha)} g(.,.)](x,y) \tag{12}$$

Moreover,

$$(\partial^{f})^{(\alpha)} P_t^{f}[g(.,.)](x,y) - (\partial^{f_n})^{(\alpha)} P_t^{f_n}[g(.,.)](x,y) \tag{13}$$

is solution of the problem

$$\varphi_0 = ((\partial^f)^{(\alpha)} - (\partial^{f_n})^{(\alpha)})g(.,.)\,;\ \frac{\partial}{\partial t}\varphi_t = \Delta^f \varphi_t + (\Delta^f - \Delta^{f_n})\varphi_t^n \tag{14}$$

where $\varphi_t^n = P_t^{f_n}[(\partial^{f_n})^{(\alpha)} g(.,.)](.,.)$. We solve this problem by the method of variation of constant. The result comes from the fact that a function which has all its derivatives in the distributional sense in L^2 is a smooth function whose C^k uniform norm can be estimated in terms of the L^2 norm of his higher derivatives.$\diamondsuit$.

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LORENTZ-COVARIANT THEORIES OF HIGHER-SPIN FIELDS AND INSIDE

V. V. Dvoeglazov

Universidad de Zacatecas, Apartado Postal 636, Suc. 3 Cruzes
Zacatecas 98064, Zac., México

Abstract

We generalize the Stueckelberg formalism in the $(1/2, 1/2)$ representation of the Lorentz Group. We analize the problem of the mass generation and of the indefinite metrics from the modern viewpoints. Some relations to other modern-physics models are found.

I. Introduction

Recent advances in astrophysics [1] suggest the existence of fundamental scalar cosmological fields [2, 3]. On the other hand, the $(1/2, 1/2)$ representation of the Lorentz group provides suitable frameworks for introduction of the $S = 0$ field, Ref. [4]. In a series of papers, starting from the very beginning we propose a generalized theory in the 4-vector representation, for the antisymetric tensor field of the second rank as well [5], see also [6]. The results can be useful in any theory dealing with the light phenomena and vector bosons. The plan of my talk is following:

- Antecedents. Mapping between the Weinberg-Tucker-Hammer (WTH) formulation and antisymmetric tensor (AST) fields of the 2nd rank. Modified Bargmann-Wigner (BW) formalism. Pseudovector potential. Parity.

- Matrix form of the general equation in the $(1/2, 1/2)$ representation.
- Lagrangian in the matrix form. Masses.
- Standard Basis and Helicity Basis.
- Dynamical invariants. Field operators. Propagators.
- The indefinite metric.
- The Gelfand-Tsetlin-Sokolik-type quantum field theory.
- The Spin-2 Framework.
- Non-commutativity.

II. Results and Conclusions

- The mapping exists between the Weinberg-Tucker-Hammer (WTH) formalism for $S = 1$ and the antisymmetric tensor fields (AST) of four kinds (provided that the solutions of the WTH equations are of the definite parity).
- Their massless limits contain additional solutions comparing with the Maxwell equations. This was related to the possible theoretical existence of the Ogievetskiĭ-Polubarinov-Kalb-Ramond notoph, Ref. [7].
- In some particular cases ($A = 0, B = 1$, see ref. [5]) the massive solutions of different parities are naturally divided into the classes of causal and tachyonic solutions.
- If we want to take into account the solutions of the WTH equations of different parity properties, this induces us to generalize the Bargmann-Wigner, Proca and the Duffin-Kemmer formalisms.
- In the $(1/2, 0) \oplus (0, 1/2)$, $(1, 0) \oplus (0, 1)$ etc. representations it is possible to introduce the parity-violating frameworks. The corresponding solutions are the mixing of various polarization states.

- The sum of the Klein-Gordon equation with the $(S, 0) \oplus (0, S)$ equations may change the theoretical content even on the free level. For instance, the higher-spin equations may actually describe various spin and mass states.

- The mappings exists between the WTH solutions of undefined parity and the AST fields, which contain both tensor and dual tensor. They are eight.

- The 4-potentials and electromagnetic fields [8, 9] in the helicity basis have different parity properties comparing with the standard basis of the polarization vectors.

- In the previous paper [10] and several talks I presented the theory in the $(1/2, 0) \oplus (0, 1/2)$ representation in the helicity basis. Under the space inversion operation, different helicity states transform each other, $Pu_h(-\mathbf{p}) = -iu_{-h}(\mathbf{p})$, $Pv_h(-\mathbf{p}) = +iv_{-h}(\mathbf{p})$.

- So, from the abovementioned (an my previous papers) it is not difficult to understand the importance of $\tilde{A}_\mu \sim \partial_\mu \chi$ term in the electrodynamics and in the Proca theory, cf. [11].

- The $(1/2, 1/2)$ representation contains both the spin-1 and spin-0 states (cf. with the Stueckelberg formalism).

- Unless we take into account the fourth state (the "time-like" state, or the spin-0 state) the set of 4-vectors is *not* a complete set in a mathematical sense.

- We cannot remove terms like $(\partial_\mu B^*_\mu)(\partial_\nu B_\nu)$ terms from the Lagrangian and dynamical invariants unless we apply the Fermi method, i. e., manually. The Lorentz condition applies only to the spin-1 states.

- We have some additional terms in the expressions of the energy-momentum vector (and, accordingly, those of the 4-current and the Pauli-Lunbanski vectors), which are the consequence of the impossibility to apply the Lorentz condition for spin-0 states.

- The helicity vectors are not the eigenvectors of the parity operator. Meanwhile, the parity is a "good" quantum number, $[\mathcal{P}, \mathcal{H}]_- = 0$ in the Fock space.

- We are able to describe the states of different masses in *any* group representation from the beginning.

- Various-type field operators can be constructed in the $(1/2, 1/2)$ representation space. For instance, they can contain C, P and CP conjugate states. Even if $b_\lambda^\dagger = a_\lambda^\dagger$ we can have complex 4-vector fields. We found the relations between creation, annihilation operators for different types of the field operators B_μ.

- Propagators have good behavious in the massless limit as opposed to those of the Proca theory. In teh generalized Stueckelberg theory one should follow the method developed in ref. [12].

The detailed explanations of several claims presented in this talk are given in journal publications. I am grateful to Profs. V. Gusynin, M. Khlopov, Y. S. Kim, M. Kirchbach, S. I. Kruglov, D. J. Cirilo-Lombardo, N. Mankoc-Borstnik, H. B. Nielsen, W. Rodrigues, R. Yamaleev, and participants of the recent conferences for useful discussions.

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FINITE CHARGE AND MASS RENORMALIZATION IN QUANTUM ELECTRODYNAMICS

I. D. Feranchuk , S. I. Feranchuk

Belarusian State University, 220030 Minsk, Belarus

Abstract

The self-localized quasi-particle excitation of the electron-positron field is found for the first time in the framework of a standard form of the quantum electrodynamics. This state is interpreted as the "physical" electron (positron) and it leads to the perturbation theory being free from the ultraviolet divergence.

I. Introduction

It is no doubt at present that the Standard Model is the fundamental basis for the theory of the electro-weak interaction [1]. It means that the quantum electrodynamics (QED) is actually the part of the general gauge theory. Nevertheless, QED considered by itself as the isolated system remains the most successful quantum field model that allows one to calculate the observed characteristics of the electromagnetic processes with a unique accuracy (for example, [2]). It is well known that these calculations are based on the series of rules connected with the perturbation theory in the observed charge e of the "physical" electron and the renormalization property of QED. The latter one means that the "primary" parameters of the theory (the charge e_0 and the mass m_0 of the "bare" electron), that are defined by the divergent integrals, can be excluded from the observed values. However, even the creators of the present form of QED were not satisfied with these rules [3](§81), [4]. It is also very essential that the dynamical description of the internal structure of the "physical" electron gives the fundamental possibility to consider muon as an excited state of the electron-positron field as it has been shown by Dirac [5].

The relation between the "primary" coupling constant e_0 and the charge e is undetermined in the present form of QED. Therefore it is possible that the value e_0 is large in spite the observed renormalized charge is small $e << 1$. Our main goal is to find such a form of the renormalization that would be logically consistent but the calculation possibilities of QED for the observed values would be preserved.

II. Construction of the self-localized state

It is well known that the spatially localized states are very important for a lot of quantum field models. Let us now consider the nonperturbative analysis of the spectrum of the one-particle excitations of the QED Hamiltonian that is defined by the following form (for example, [6]) :

$$\hat{H} = \int d\vec{r} : \{\hat{\psi}^*(\vec{r})[\vec{\alpha}(\vec{p} + e_0\hat{\vec{A}}(\vec{r})) + \beta m_0]\hat{\psi}(\vec{r}) + e_0\hat{\varphi}(\vec{r})\hat{\rho}(\vec{r}) -$$

$$-\frac{1}{2}(\vec{\nabla}\hat{\varphi}(\vec{r}))^2\} : + \sum_{\vec{k}\lambda} \omega(\vec{k})\hat{n}_{\vec{k}\lambda}; \quad \hat{\rho}(\vec{r}) = \frac{1}{2}[\hat{\psi}^*(\vec{r})\hat{\psi}(\vec{r}) - \hat{\psi}(\vec{r})\hat{\psi}^*(\vec{r})]. \qquad (1)$$

We suppose here that the field operators are given in the Schrödinger representation, the spinor components of the electron-positron operators being defined in the standard way [6].

In these formulas $\hbar = c = 1$; the primary charge $(-e_0), e_0 > 0$ and m_0 are considered as the parameters of the model; the symbol $: \hat{H} :$ means the normal ordering of the operators excluding the vacuum energy [3]; $\vec{\alpha}, \beta$ are Dirac matrixes; $a_{\vec{p}s}(a^+_{\vec{p}s})$ and $b_{\vec{p}s}(b^+_{\vec{p}s})$ are the annihilation (creation) operators for the "bare" electrons and positrons in the corresponding states. The field operator $\hat{\vec{A}}(\vec{r})$ and the operator of the photon number $\hat{n}_{\vec{k}\lambda}$ are related to the transversal electromagnetic field.

For the variational description of the self-consistent excitation let us choose the trial state vector $|\Phi_1>$ in the general form of the wave packet formed by the one-particle excitations of the "bare" electron-positron field depending on the set of variational classical functions $U_{\vec{q}s}; V_{\vec{q}s}; \varphi(\vec{r})$. Besides, the effect of polarization and the appearance of the electrostatic field $\varphi(\vec{r})$ should be taken into account, so we consider $|\Phi_1>$ to be the eigenvector for the operator of the scalar field:

$$|\Phi_1 > \simeq |\Phi_1^{(0)}(U_{\vec{q}s}; V_{\vec{q}s}; \varphi(\vec{r})) >= \int d\vec{q}\{U_{\vec{q}s} a^+_{\vec{q}s} + V_{\vec{q}s} b^+_{\vec{q}s}\}|0; 0; \varphi(\vec{r}) > . \quad (2)$$

The ground state of the system is $|\Phi_0 >= |0; 0; 0 >$, if we use the same notation. It corresponds to the vacuum of both interacting fields.

Firstly, let's consider the excitation with the zero total momentum. Then the constructed trial vector should satisfy the normalized conditions resulting from the definition of the total momentum $\vec{P}$ and the observed charge e of the "physical" particle:

$$< \Phi_1^{(0)}|\hat{\vec{P}}|\Phi_1^{(0)} >= \sum_s d\vec{q}\vec{q}[|U_{\vec{q}s}|^2 + |V_{\vec{q}s}|^2] = \vec{P} = 0;$$

$$\sum_s d\vec{q}[|U_{qs}|^2 + |V_{qs}|^2] = 1;$$

$$< \Phi_1^{(0)}|\hat{Q}|\Phi_1^{(0)} >= e_0 \sum_s d\vec{q}[|V_{qs}|^2 - |U_{qs}|^2] = e. \quad (3)$$

The last equation defines the observed charge of the "physical" particle at the given value e_0 of the initial charge of the "bare" particle. The trial vector $|\Phi_1 >$ is actually the collective excitation of the system and in this respect the variational approach differs greatly from the perturbation theory. where the zero approximation for a one-particle state correspond to one-particle excitations determined by the charge e_0 of the "bare" electron and the field $\varphi(\vec{r}) = 0$.

Thus, the following variational estimation for the energy $E_1(0)=E_1(\vec{P}=0)$ of the state corresponding to the "physical" quasi-particle excitation of the whole system :

$$E_1(0) \simeq E_1^{(0)}[U_{qs}; V_{qs}; \varphi(\vec{r})] =< \Phi_1^{(0)}|\hat{H}|\Phi_1^{(0)} >, \quad (4)$$

where the average is calculated with the full Hamiltonian (1) and the functions $U_{qs}; V_{qs}; \varphi(\vec{r})$ are to be found as the solutions of variational equations with the additional conditions (3) .

The average value in Eq. (4) is calculated neglecting the classical components of the vector field. They are appeared in the high-order corrections that are defined by the renormalized charge $e1$ and can be considered by means of the canonical perturbation theory. It should be noted that the possibility of constructing self-consistently the renormalized QED at the non-zero vacuum value of the scalar field operator was considered before [7] but the solution of the corresponding equations was not discussed.

In order to vary the introduced functional let us define the spinor wave functions (not operators) which describe the coordinate representation for the electron and positron wave packets in the state vector $|\Phi_1^{(0)}>$:

$$\Psi_\nu(\vec{r}) = \int \frac{d\vec{q}}{(2\pi)^{3/2}} \sum_s U_{qs} u_{\vec{q}s\nu} e^{i\vec{q}\vec{r}}; \ \Psi_\nu^c(\vec{r}) = \int \frac{d\vec{q}}{(2\pi)^{3/2}} \sum_s V_{qs}^* v_{\vec{q}s\nu} e^{i\vec{q}\vec{r}}. \quad (5)$$

Varying the functional (4) by the wave functions $\Psi(\vec{r})$ and $\Psi^c(\vec{r})$ taking into account their normalization conditions one can find the equivalent Dirac equations describing the electron (positron) motion in the field of potential $\varphi(\vec{r})$:

$$\begin{gathered}
\{(-i\vec{\alpha}\vec{\nabla} + \beta m_0) + e_0\varphi(\vec{r})\}\Psi(\vec{r}) = 0; \\
\{(-i\vec{\alpha}\vec{\nabla} + \beta m_0) + e_0\varphi(\vec{r})\}\Psi^c(\vec{r}) = 0, \\
\varphi(\vec{r}) = \frac{e_0}{4\pi} \int \frac{d\vec{r}'}{|\vec{r} - \vec{r}'|}[\Psi^+(\vec{r}')\Psi(\vec{r}') - \Psi^{+c}(\vec{r}')\Psi^c(\vec{r}')]. \quad (6)
\end{gathered}$$

But it is important that in spite of the normalization condition (3) for the total state vector (5) each of its components could be normalized differently

$$\int d\vec{r}\Psi^+(\vec{r})\Psi(\vec{r}) = \frac{1}{1+C}; \ \int d\vec{r}\Psi^{+c}(\vec{r})\Psi^c(\vec{r}) = \frac{C}{1+C}. \quad (7)$$

The constant C is an arbitrary value up to now. It defines the ratio of two charge states in the considered wave packet. As a result the self-consistent potential $\varphi(\vec{r})$ of the scalar field depends on C.

Since the considered physical system has no preferred vectors if $\vec{P} = 0$, it is natural to regard the self-consistent potential as spherically symmetrical. Then the variable separation for the Dirac equation is realized on the basis of the well known spherical bispinors [2]. Then the unknown radial functions f, g satisfy the following system of the equations:

$$\begin{aligned} &\frac{d(rg)}{dr} - \frac{1}{r}(rg) - (m_0 - e_0\varphi(r))(rf) = 0; \\ &\frac{d(rf)}{dr} + \frac{1}{r}(rf) - (m_0 + e_0\varphi(r))(rg) = 0. \end{aligned} \tag{8}$$

The equations for the radial wave functions f_1, g_1 of the positron components:

$$\begin{aligned} &\frac{d(rg_1)}{dr} + \frac{1}{r}(rg_1) - (m_0 + e_0\varphi(r))(rf_1) = 0; \\ &\frac{d(rf_1)}{dr} - \frac{1}{r}(rf_1) - (m_0 - e_0\varphi(r))(rg_1) = 0. \end{aligned} \tag{9}$$

The equation for the self-consistent potential follows from the definition of $\varphi(r)$ taking into account the normalization of the spherical spinors [2]:

$$\frac{d^2\varphi}{dr^2} + \frac{2}{r}\frac{d\varphi}{dr} = -\frac{e_0}{4\pi}[f^2 + g^2 - f_1^2 - g_1^2]. \tag{10}$$

The boundary condition for the potential defines the charge e of the "physical" electron (positron)

$$\varphi(r)|_{r\to\infty} = \frac{e}{4\pi r} = \frac{e_0}{4\pi r}\int_0^\infty r_1^2 dr_1[f^2(r_1) + g^2(r_1) - f_1^2(r_1) - g_1^2(r_1)]. \tag{11}$$

The structure of the equation (6) shows that the considered variational method is consistent with the gauge symmetry of the initial Hamiltonian. One can show that the Hamiltonian (1) could be chosen in an arbitrary Lorentz gauge with the classical components both for the scalar field $\varphi(\vec{r})$ and for the longitudinal field $\vec{A}_l(\vec{r})$ [8].

Dimensionless variables and new functions can be introduced

$$\begin{aligned} &x = rm_0; E = \epsilon m_0; e_0\varphi(r) = m_0\varphi(x); \frac{e_0^2}{4\pi} = \alpha_0; u(x)\sqrt{m_0} = rg(r); \\ &v(x)\sqrt{m_0} = rf(r); u_1(x)\sqrt{m_0} = rg_1(r); v_1(x)\sqrt{m_0} = rf_1(r). \end{aligned} \tag{12}$$

As a result the system of equations for describing the radial wave functions of the one-particle excitation of the electron-positron field and the self-consistent potential of the vacuum polarization can be obtained:

$$u,v=\sqrt{\frac{1}{1+C}}u_0,v_0;u_1,v_1=\sqrt{\frac{C}{1+C}}v_0,u_0;$$
$$\int_0^\infty dx[u_0^2(x)+v_0^2(x)]=1;\rho_0(x)=u_0^2(x)+v_0^2(x);$$
$$\frac{du_0}{dx}-\frac{1}{x}u_0-(1-\varphi(x))v_0=0;\frac{dv_0}{dx}+\frac{1}{x}v_0-(1+\varphi(x))u_0=0;$$
$$\varphi(x)=\alpha_0\frac{1-C}{1+C}\varphi_0(x);\varphi_0(x)=[\int_x^\infty dy\frac{\rho_0(y)}{y}+\frac{1}{x}\int_0^x dy\rho_0(y)]. \qquad (13)$$

The energy of the system can also be calculated with these functions:

$$E_1(0)\equiv E(0)=m_0\frac{1-C}{1+C}[T+\frac{1}{2}\alpha_0\frac{1-C}{1+C}\Pi];$$
$$T=\int_0^\infty dx[(u_0'v_0-v_0'u_0)-2\frac{u_0v_0}{x}+(u_0^2-v_0^2)];$$
$$\Pi=\int_0^\infty dx\varphi_0(x)(u_0^2+v_0^2). \qquad (14)$$

and Eq.(13) can be obtained when varying of the functional (14).

The value $a=\alpha_0(1-C)/(1+C)$ is the free parameter of the equations (13) and it plays a role of the eigenvalue when the nontrivial normalized solution exists.

The method for the numerical solution of the nonlinear self-consistent system of the equations (13) was described in detail in the paper [8]. Only the numerical results for the localized wave functions and for the scalar potential are described in the present work. The numerical value for the parameter a depends on the accuracy of the finite-difference approximation for the differential operators and was as $a=a_0\approx-3.531$.

The solutions u_0,v_0 for the electron and positron components and the self-consistent potential were drawn in Ref. [8]. All these functions are localized in the domain with the linear size of $\sim m_0^{-1}$. The potential gets over the Coulomb potential of the "physical" charge e for $r>r_0=m_0^{-1}$.

It is important that the characteristic size of this excitation r_0 is the same order as the classical radius of the electron $r_e = \alpha/m$, namely $r_0 = \frac{r_e}{2|a_0|} \approx 0.15 r_e$.

The stationary localized collective excitation of the electron-positron field described above is of great interest by itself as the eigenvector of the well known QED Hamiltonian that can't be calculated by means of the perturbation theory and has not be considered before. It is naturally to suppose that this localized state describes the "physical" electron (positron) with the observed charge e. The integral charge of the considered one-particle excitation is defined by the boundary condition (11) and this supposition leads to $e_0(1-C)/(1+C) = e$. In the result one can find the following relation between the "primary" coupling constant $\alpha_0 = e_0^2/4\pi$ and the observed value of the fine structure constant $\alpha = e^2/4\pi$

$$\alpha_0 = \frac{a_0^2}{\alpha} \approx 1708.1. \tag{15}$$

This formula defines the renormalization of the charge in the considered approximation and shows self-consistency of the initial supposition that the interaction between the "primary" electron-positron and scalar fields is strong.

Then the total energy of the excitation with zero momentum is:

$$E(0) = -\frac{m_0}{\alpha_0}\frac{T a_0}{2} = -m_0 \alpha \frac{T}{2a_0} > 0. \tag{16}$$

This value defines the minimal energy of the one-particle excitation of the electron-positron field and its positive sign corresponds to the "bottom" of the "physical" electron zone in the renormalized QED. It was also shown in [8] that $E(0)$ can be considered as the "physical" electron mass m_e because it defines the spectrum of the excitation with non zero total momentum $\vec{P}$ by Lorentz invariant way:

$$E(\vec{P}) = \sqrt{P^2 + E^2(0)};\ E(0) \equiv m_e = -m_0 \alpha \frac{T}{2a_0};$$
$$m_0 = m_e \frac{2|a_0|}{\alpha} \approx 1291.7 m_e. \tag{17}$$

As it was shown by Dirac [5], investigation of the "physical" electron with the distributed charge gives the possibility to interpret the "physical" muon as the excited state of such system. The variational approach considered in the present paper allows one to analyze the one-particle excitation differed from the "physical" electron without inclusion of any additional parameters.This approach leads to a quite reasonable estimation [8] for muon mass $(m_\mu/m_e) \approx 191$ instead of the experimental value $(m_\mu/m_e)_{exp} \approx 206$.

It was also shown in [8] that the interaction between the considered "physical" electron and the transversal electromagnetic field corresponds to the perturbation theory relatively to the "physical" charge $e l1$ but without the divergent integrals.

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SINGULAR LOCALIZED STATES, EXACTLY SOLVABLE AND QUASI-EXACTLY SOLVABLE PROBLEMS IN STOCHASTIC DYNAMICS

G. Krylov

Physics Department, Belarusian State University, 4 Nezavisimosti av., Minsk, 220030 Belarus

Abstract

Based on SUSY QM approach to stochastic problems the construction of exactly solvable and quasi-exactly solvable problems is considered. The possibility of existence of singular localized eigenstates for linear Fokker-Planck equation has been explicitly demonstrated.

I. Introduction

The problem of construction of new exactly solvable problems continues to be very attractive. Even for cases of the most developed one dimensional quantum mechanics, where there are about 50 known solvable potentials (see, e.g. [1]) as well as a large number of quasi-exactly solvable ones, new papers attacking the problem appears every month.

At the same time only few examples of solvable Fokker-Planck equations are known. One of the goals of the paper will be to consider the problem of solubility of the last equation based on SUSY QM approach.

Another interesting question in the field is related to the so called "blow up regimes" for some nonlinear equations. One of the first report of this phenomenon for quasi-linear heat transfer equation has been written in the middle of 80-th by A. Samarski et al. [2]. They found that for the equation

$$\partial_t f(x,t) = \partial_x (D(f)\partial_x f(x,t)) + U(f,x,t) \tag{1}$$

with a nonlinear heat transfer coefficient D depending on temperature (f) as $D(f) \sim f^\sigma,\ \sigma > 1$ and for source functions of the form $U(f,x,t) = bf^\rho$, the existence of new regimes is possible (the so called "blowup", "heat explosion" and "heat localization" regimes) when singularity of f is produced within the finite time interval. Later such regimes have attracted much attention in different fields see e.g., [3] and bibliography therein.

It is commonly accepted that singular localization is an inevitably nonlinear effect, typically originated from the existence of some generalized symmetry and therefore some self-similar solutions.

The second goal of the paper is to demonstrate that singular localized solutions (eigenstates) can naturally appear in some linear problems for the Fokker-Planck (F-P) equation in an external field.

The paper is organized as follows. In the second section we shortly outline the correspondence between Shrödinger and Fokker-Planck equations arisen within the framework of supersymmetric quantum mechanics (SUSY QM) to stochastic problems [4, 6] and construct a series of solvable potentials. In the third section we use one quantum quasi-exactly solvable problem and construct the appropriate F-P problem with singular localized eigenfunctions.

II. Correspondence between the Shrödinger and Fokker-Plank equations

One dimensional diffusion equation for the distribution function $f(x,t)$ for a system in an external field with a potential $U(x)$ reads

$$\frac{\partial f(x,t)}{\partial t} = D\frac{\partial^2 f(x,t)}{\partial x^2} + a\frac{\partial}{\partial x}\left(f(x,t)\frac{dU(x,t)}{dx}\right), \tag{2}$$

where D is the diffusion coefficient, a is the coupling constant for interaction of a particle with an external potential U(x). In subsequent we incorporate it directly to the potential putting $a = 1$.

In the literature it is commonly accepted that the only difference of diffusion equation and Schrödinger's one is in imaginary time on respect to real time (Vick's rotation). Though it is evidently true for the case of a free particle, for the problem in an external field the only sight on the second equation

$$i\hbar\frac{\partial\psi(x,t)}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}+U(x)\psi(x,t) \tag{3}$$

immediately demonstrates that the external field is incorporated into the equation (3) in a different way with respect to that in the diffusion case (2).

The prominent feature of the eq.(2) is the existence of zero-mode (stationary or steady-state) solution $f_s(x)$, which simply corresponds to the known Boltzmann distribution $f_s(x)=C\exp\left(-U(x)\right)$.

In opposite, for the Schrödinger equation (3) the ground state is typically unknown and of most interest.

This, as we will see, is due to the fact that after transformation of the diffusion equation into the form of the Schrödinger one, we obtain the last in the supersymmetric quantum mechanics (SUSY) form directly and the proper partner Hamiltonian is just H_- [5].

Let us shortly outline this way [4, 6]. We assume the units' choice is such that $\hbar=1, m=1, D=1/2$. It is worth to note that the steady state solution reads $f_s(x)=\exp(-2U(x))$ with this units' choice.

Then, after substitution $f(x)=\exp\{-U(x)-Et\}\,\psi(x)$ into

$$\frac{\partial f(x,t)}{\partial t}=\frac{1}{2}\,\frac{\partial^2 f(x,t)}{\partial t^2}+\frac{\partial}{\partial x}\left(f(x,t)U'(x)\right) \tag{4}$$

we get the Schrödinger equation in the form

$$\frac{1}{2}\psi''(x)+\left(E-V_q(x)\right)\psi(x)=0 \tag{5}$$

with a "quantum potential" $V_q(x)$ given by

$$V_q(x)=\frac{1}{2}\,U'(x)^2-\frac{1}{2}\,U''(x). \tag{6}$$

The last equation is just in the form of SUSY QM approach with the *superpotential* given by $W(x)=U'(x)$ [4] and the Hamiltonian operator having the factorized form

$$\hat{H}_- = \hat{A}^\dagger \hat{A} = \frac{1}{\sqrt{2}} \left(-\frac{d}{dx} + U'(x) \right) \frac{1}{\sqrt{2}} \left(\frac{d}{dx} + U'(x) \right). \tag{7}$$

It is evident from the (5) and (7) that the state $E = 0$ is the eigenstate of H_-.

One can exploit the supersymmetric form directly by the construction of solvable cases for 1-D diffusion equation, considering known shape-invariant partner potentials [4].

There is another way, namely to construct the superpotential $W(x) = U'(x)$ that leads to exactly-solvable potentials for eq.(5). Denoting a solvable quantum potential in (5) by $V_s(x)$, we consider eq.(6) as the Ricatti equation for the superpotential $W(x)$

$$W'(x) - W(x)^2 = -2\, V_s(x). \tag{8}$$

Here it is worth to point out that we can split the energy parameter E in (5) as $E = E_1 + E_2$ that leads to the appearance of a term e.g., E_2 in the right side of (8) and can be convenient in subsequent.

Based on the known correspondence of Ricatti and Schrödinger equations we make substitution

$$W(x) = -\Psi'(x)/\Psi(x)) \tag{9}$$

and rewrite (8) in the form of the Schrödinger equation for the function $\Psi(x)$

$$\frac{1}{2}\Psi''(x) + (E_2 - V_s(x))\, \Psi(x) = 0. \tag{10}$$

The last equation means that every eigenstate $\Psi_n(x)$ of a quantum solvable potential $V_s(x)$ gives a superpotential through the relation (9) that after integration gives the diffusion equation potential $U(x)$ in the form

$$U_n(x) = U_0 + \log |\Psi_n(x)| \,. \tag{11}$$

It is interesting that the set $U_n(x)$ leads to the same Schrödinger equation (5) (with different splitting of the constant E).

The i-th eigenstate for the exactly solvable diffusion problem with the potential $U_n(x)$ reads

$$f_i(x,t) = \Psi_n(x)\exp\left(-(E_{i+n} - E_n)t\right)\Psi_{i+n}(x), i = 0, 1, ... \tag{12}$$

where E_i is eigenenergy of the appropriate quantum potential).

We can construct examples of exactly-solvable diffusion potential using eq.(11) and, e.g., known solutions for the quantum harmonic oscillator. Its potential is $V_s(x) = x^2/2$, the eigenfunctions read (omitting normalization factor)

$$\Psi_n(x) = H_n(x)\exp(-x^2/2) \quad n = 0, 1... \tag{13}$$

where $H_n(x)$ are Hermite polynomials and eigenenergies are given by $E_n = n + 1/2$. The diffusion case (potential,ground eigenstate and the F-P equation) reads

$\underline{n = 0}$

$$U_0(x) = \frac{x^2}{2}, \qquad f_0(x) = e^{-x^2}$$

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 f(x,t)}{\partial x^2} + x\,\frac{\partial f(x,t)}{\partial x} + f(x,t),$$

$\underline{n = 1}$

$$U_1(x) = \frac{x^2}{2} - \log|x|, \qquad f_0(x) = x^2\,e^{-x^2}$$

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2}\,\frac{\partial^2 f(x,t)}{\partial x^2} + \left(x - \frac{1}{x}\right)\frac{\partial f(x,t)}{\partial x} + \left(1 + \frac{1}{x^2}\right)f(x,t),$$

$\underline{n = 3}$

$$U_3(x) = \frac{x^2}{2} - \log|x(2x^2 - 3)|, \qquad f_0(x) = x^2\,(2\,x^2 - 3)^2 e^{-x^2}$$

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2}\,\partial^2 f(x,t)\partial x^2 + \left(x - \frac{1}{x} - \frac{4x}{2x^2 - 3}\right)\frac{\partial f(x,t)}{\partial x} +$$

$$\left(1+\frac{1}{x^2}+\frac{16x^2}{(2x^2-3)^2}-\frac{4}{2x^2-3}\right)f(x,t).$$

We demonstrate the diffusion potential and first two eigenstates for the case $n = 3$ in two adjoined wells, ($x \in [0, \sqrt{3/2}]$ and $x \in [\sqrt{3/2}, \infty]$) in Fig. 1.

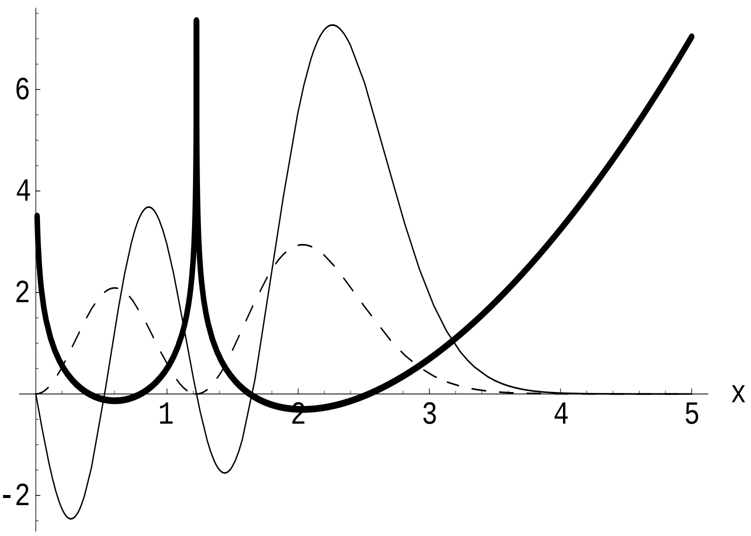

Figure 1: Diffusion equation potential $U_3(x)$ (bold solid line) and first two eigenstates $f_0(x), f_1(x)$ (solid, and dashed lines, non-normalized) in two adjoined infinite barrier wells.

The constructed solvable potentials are logarithmically singular, so the question arises either they correspond to non-penetrable multi-wall diffusion problem, or diffusion takes place in all space. The question needs more deep investigation but first conclusion is that such walls are partially penetrable within the ordinary diffusion model that ignores particle momentums (and we can see e.g., non-zero slopes for higher eigenstates functions in Fig. 1).

III. Singular localized eigenstates for the F-P equation

In fact, the method outlined in the previous section is not restricted to the construction of exactly solvable diffusion models only, it could be also used and for the quasi-exactly solvable ones. The last are such systems that allow algebraic construction of only a finite number of eigenstates (see e.g., [7] for more detail and ref. therein). Let us use it for the explicit construction of singular localized eigenstates for the F-P equation.

In the paper [8, 9] the method has been proposed for the construction of 1D solvable and quasi-solvable potential families in QM based on polynomial Ansatz for the wave function.

The general form of the second order linear differential equations allowing polynomial solutions at some specifically chosen values of their coefficients reads [8]

$$\hat{\mathcal{L}}_k y(x) = P_{k+2}(x) y''(x) + Q_{k+1}(x) y'(x) + R_k(x) y(x) = 0. \tag{14}$$

It is easily understood that differential operator $\hat{\mathcal{L}}_k$ maps the space of the n-th order polynomials $F_n[x]$ into the space $F_{n+k}[x]$. As both spaces are finite dimensional, the condition of non-trivial kernel $Ker\hat{\mathcal{L}} \neq 0$ leads simply to a linear algebraic problem for operator representation in this space plus k additional conditions imposed on the coefficients of coefficient functions.

One example we discussed in [9] was

$$x^3 y''(x) + \alpha(x^2 - 1) y'(x) + (\beta x + \gamma) y(x) = 0. \tag{15}$$

The Schrödinger equation

$$Y''(u) + (\varepsilon - V(u))\, Y(u) = 0 \tag{16}$$

for this case has the potential $V(u)$ of the form

$$V(u) = \frac{A}{u^2} + Bu^2 + Cu^4 + Du^6. \tag{17}$$

Explicit formulae for the coefficients A, B, C, D can be found it [9].

Polynomial Ansatz for $n = 1$ leads to two eigenstates with the energies $\epsilon = \pm\alpha$ and eigenfunctions given as

$$Y^{(0)}(u) = \exp\left\{\frac{\alpha u^4}{64}\right\}\left(\frac{4}{u^2}+1\right)u^{\frac{3-2\,\alpha}{2}}, \tag{18}$$

$$Y^{(1)}(u) = \exp\left\{\frac{\alpha u^4}{64}\right\}\left(\frac{4}{u^2}-1\right)u^{\frac{3-2\,\alpha}{2}}. \tag{19}$$

The "admissible region" for the parameter α is given by $\alpha \leq -1/4$ (so that the eigenfunction is square integrable and non-singular). Then the constructed eigenstates represent the ground and the first excited states for the potential

$$V(u) = \frac{\alpha^2 u^6}{256} - \frac{\alpha\,(\alpha-3)\,u^2}{8} + \frac{4\alpha^2+24\,\alpha+35}{4\,u^2}. \tag{20}$$

If one considers the region $-1/4 \leq \alpha < 0$, it is easily checked that the eigenfunctions in (18,19) have integrable singularity at $x = 0$.

The substitution of the explicit formula (18) gives for the diffusion potential

$$U(x) = U_0 + -\alpha x^4 \log\left(x^{-1/2-\alpha}\,\left(4+x^2\right)\right), \tag{21}$$

for the steady state eigenfunction

$$f_0(x,t) = \left(Y^{(0)}(u)\right)^2 = \left(\exp\left\{\frac{\alpha u^4}{64}\right\}\left(\frac{4}{u^2}+1\right)u^{\frac{3-2\,\alpha}{2}}\quad\right)^2 \tag{22}$$

and the first excited state eigenfunction

$$f_1(x,t) = Y^{(0)}(x)Y^{(1)}(x) = e^{2\,\alpha x^4-\alpha t}\left(1-\frac{16}{x^4}\right)x^{3-2\,\alpha}. \tag{23}$$

In Fig. 2 we demonstrate the quantum and diffusion potential and ground and the first excited eigenstates for $\alpha = -1/4$ with evident singularity at $x = 0$.

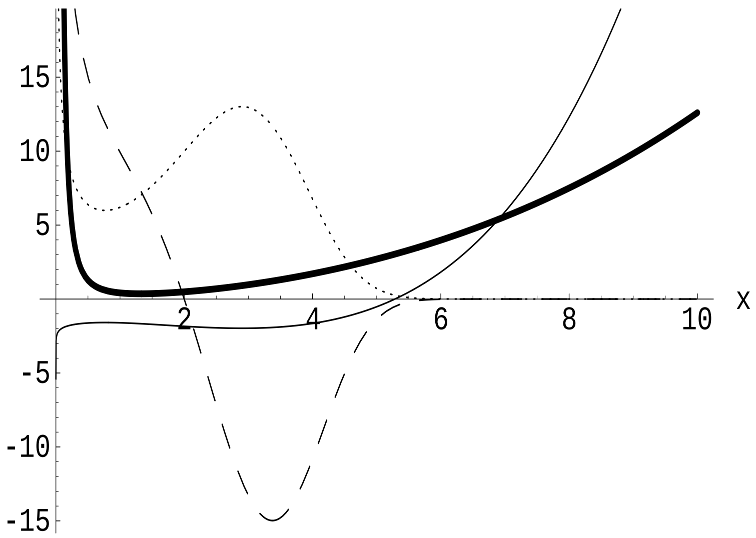

Figure 2: Quantum and diffusion potentials (bold and solid lines), steady state and the first excited state (dotted and dashed lines) for $\alpha = -1/4$.

The obtained result allows us to say that we indeed constructed singular localized states for the Fokker-Planck (diffusion) equation, that can be considered as linear analogs of "heat localization" regimes known in the theory of quasi-linear equations.

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SECONDARY QUANTIZED PROBLEM OF PAIR CREATION: PROJECTION OPERATOR TECHNIQUE

H. V. Krylova

Physics department, Belarusan State University
4 Nezavisimosti Ave., 220030 Minsk, the Republic of Belarus

Abstract

In the work a secondary quantized wave function of many fermion systems has been found in terms of one-particle fermion creation (annihilation) operators and two-particle creation (annihilation) operators. A Green function method developed has been applied for the quantum field description of the problem on pair creation.

I. Introduction

It is known that fermion pair creation appears in a large number of physical situations described in condensed matter, atomic, nuclear, elementary particle physics, astrophysics, and cosmology. Therefore, the problem of pair production from electric fields has been the subject of considerable theoretical interest [1]. A Dirac problem of pair production in a homogeneous electric field $\vec{E}$ rotating in plane has been considered in [2]. Symmetry of this Dirac problem is described by the group SO(4) [2]. The Dirac equation describes a classical fermion field. However the Dirac operator has unphysical states that leads to Klein paradox in a problem of electron scattering on a potential barrier [3]. The Dirac equation describes a motion of an electron, and its Dirac conjugation describes motion of a positron. Therefore that fact is surprising that the states belonging to the energy gap of the Dirac operator, describe a fermion pair arising in a homogeneous electric field rotating in the plane. In the secondary quantized Dirac problem of pair production in the homogeneous electric field $\vec{E}$ a secondary quantized fermion field is represented as a set of electrons and positrons, described by complex spinor which real components are electronic ones, and imaginary components are positronic ones [4]. An interaction Hamiltonian of this problem can be constructed on generators of the algebra of group SO(4) which is locally isomorphic to the group SU(2)×SU(2) [5]. To date, an analytic formalism that successfully addresses the general problem of fields which vary arbitrarily in both time and space has not been developed. The goal of the work is to offer a projection operator technique for a secondary quantized problem of pair creation.

II. Secondary quantized problem of pair production in a homogeneous electric field rotating in the plane

We can define an operator of electron creation $\widehat{\varphi_1}^+$ as quantized positively frequency part ${\varphi_1}^+$ of field function φ_1 , and an operator of positron creation $\widehat{{\varphi^\dagger}_1}^+$ as quantized positively frequency part $\varphi^{\dagger +}_1$ of Hermitian conjugate field function ${\varphi^\dagger}_1$. Accordingly, the operator of electron annihilation $\widehat{{\varphi^\dagger}_1}^-$ is defined as quantized negatively frequency part $\varphi^{\dagger -}_1$ of Hermitian conjugate field function ${\varphi^\dagger}_1$, and the operator of positron annihilation $\widehat{\varphi}_1{}^-$ quantized as negatively frequency part φ_1^- of field function φ_1 . Now we can define annihilation operators $\left(\widehat{\Phi}^\dagger{}_{pair}\right)^-$ and creation operators $\left(\widehat{\Phi}_{pair}\right)^+$ of fermion pairs as

$$\left(\widehat{\Phi}^\dagger{}_{pair}\right)^- = \Phi_- = \widehat{\varphi}_1{}^- \, \widehat{{\varphi^\dagger}_1}^- , \tag{1}$$

$$\left(\widehat{\Phi}_{pair}\right)^+ = \Phi_+ = \widehat{{\varphi^\dagger}_1}^+ \, \widehat{\varphi_1}^+ , \tag{2}$$

and an operator

$$\left(\widehat{\Phi}^\dagger{}_{pair}\right)^0 = \Phi_0 = \frac{1}{2}\left(\widehat{\varphi_1}^+ \, \widehat{{\varphi^\dagger}_1}^- - \widehat{\varphi}_1{}^- \, \widehat{{\varphi^\dagger}_1}^+\right). \tag{3}$$

Substituting into invariant Casimir operator C_2 for algebra SU(2)

$$C_2 = \frac{1}{2}(\Phi_+\Phi_- + \Phi_-\Phi_+) + \Phi_0^2 \tag{4}$$

the explicit expressions for $\Phi_\pm, \Phi_0$ (1) - (3), we find the Casimir operator as

$$C_2 = \frac{3}{4}\left(1 - \left(\widehat{\varphi_1}^+ \widehat{\varphi^\dagger}_1{}^- - \widehat{\varphi^\dagger}_1{}^+ \widehat{\varphi}_1{}^-\right)^2\right). \tag{5}$$

One get a wave function $\Psi(\vec{r}_1, \vec{r}_2)$ of fermion pair with additional coupled electron by an action of the operator $\widehat{\varphi_1}^+$ and $\left(\widehat{\Phi}_{pair}\right)^+$ on a vacuum vector $|\,0\rangle$ as

$$\Psi(\vec{r}_1, \vec{r}_2) = \left\langle \vec{r}_1\,|\hat{\Psi}|\,\vec{r}_2\right\rangle = \langle \vec{r}_1\,|\,\widehat{\varphi_1}^+\,|0\rangle \left\langle 0|\left(\widehat{\Phi}_{pair}\right)^+|\,\vec{r}_2\right\rangle, \tag{6}$$

where $\vec{r}_1$ is a radius - vector of electron with spin "up", $\vec{r}_2$ is a radius - vector of electron with spin "down". Since by virtue of state orthogonality it is possible to add the projection operator $|0><0|$ in calculations up to $\hat{I}$, the expression (6) can be transformed to the form

$$|\,\Psi\rangle = \hat{\Psi}|\,0\rangle = \widehat{\varphi_1}^+\,\hat{I}\left(\widehat{\Phi}_{pair}\right)^+|\,0\rangle\,. \tag{7}$$

Neglecting correlations, the vacuum vector $|\,0\rangle$ can be presented as a product of vacuum vectors $|\,0\uparrow\rangle$ and $|\,0\downarrow\rangle$ for states with spin "up" and "down". Hence, the expression (7) can be rewritten as

$$|\,\Psi\rangle = \hat{\Psi}|\,0\rangle = \widehat{\varphi_1}^+\,|\,0\uparrow\rangle\left(\widehat{\Phi}_{pair}\right)^+|\,0\downarrow\rangle \equiv |\,1,0\rangle\,|\,1,1\rangle\,, \tag{8}$$

where $|\,1,0\rangle$ is a state with one electron, $|\,1,1\rangle$ is a state with one electron and one positron. Since, as shown above, ket-vectors $|\,1,0\rangle$ and $|\,1,1\rangle$ are transformed on a representation of the symmetry group SU(2), the wave function $|\,\Psi\rangle$ is transformed on representation of the symmetry group SO(4). Let us evaluate a value which is accepted a Casimir operator C_4 of group SO (4) on the vector $|\,1,0\rangle\,|\,1,1\rangle$ of Fock space:

$$C_4(|\,1,0\rangle\,|\,1,1\rangle) = (C_2|\,1,0\rangle)\,|\,1,1\rangle + |\,1,0\rangle\,(C_2|\,1,1\rangle)\,. \tag{9}$$

Values of the Casimir operator C_4 (9) are eigenvalues of the operaror of squared angular momentum $\hat{J}^2$ of the state $|\,1,0\rangle\,|\,1,1\rangle$ describing the system from one electron and one pair of particle - antiparticle. We see that operators $\widehat{\varphi_1}^{+}\,\widehat{\varphi^{\dagger}}_1{}^{-}$ and $\widehat{\varphi^{\dagger}}_1{}^{+}\,\widehat{\varphi}_1{}^{-}$ are operators of occupation numbers for fermions $\hat{n}_-$ and antifermions $\hat{n}_+$:

$$\hat{n}_- = \widehat{\varphi_1}^{+}\,\widehat{\varphi^{\dagger}}_1{}^{-}, \quad \hat{n}_+ = \widehat{\varphi^{\dagger}}_1{}^{+}\,\widehat{\varphi}_1{}^{-}. \tag{10}$$

Substituting the expressions (5) and (10) in the formula (9) we get $C_4 = \frac{3}{4}$ as for the state $|\,1,1\rangle$ we have

$$C_2|\,1,1\rangle = \frac{3}{4}[1 - (\hat{n}_-\,|\,1,1\rangle - \hat{n}_+\,|\,1,1\rangle)] = \frac{3}{4}, \tag{11}$$

and for the state $|\,1,0\rangle$ $C_2 = 0$ owing to identity

$$C_2|\,1,0\rangle = \frac{3}{4}(1 - \hat{n}_-\,|\,1,0\rangle + \hat{n}_+\,|\,1,0\rangle) = 0. \tag{12}$$

It means that the state $|\,1,0\rangle$ $|\,1,1\rangle$ is transformed on a spinor representation of the group SU(2). This result is an appearance of a cyclic symmetry of many fermion systems, meaning, that by virtue of identity of electrons there are configurations which are produced by a cyclic permutation from a configuration with one unpaired electron including a configuration with a "hole" - positron and electron with spin "down". Further we shall develop a technique of projection operators allowing the secondary quantization of a system with variable number of particles and pairs of particle - antiparticle.

III. Secondary quantized wave function of a system with variable number of electron and fermionic pairs

Let us consider a quantum system consisting of variable (very large) number N of identical interacting particles $N \to \infty$. Its description will be complete if one knows accurate within phase multiplier $\exp(\imath\theta)$ a vector of state $|\varphi_1>$ for one particle, a two-dimensional vector of state $|\varphi_1, \varphi_2>$ for a subsystem from two particles, a three-dimensional vector of state $|\varphi_1, \varphi_2, \varphi_3>$ for a subsystem from three particles, etc. A wave function $|\widehat{\varphi}>$ of all many particle system is described by vectors with coordinates $<\widehat{\varphi}|\varphi_0, \varphi_1, \ldots, \varphi_n>$ [6]:

$$\begin{pmatrix} <\widehat{\varphi}|\varphi_0><\varphi_0| \\ <\widehat{\varphi}|\varphi_0,\varphi_1><\varphi_0,\varphi_1| \\ \cdots \\ <\widehat{\varphi}|\varphi_0,\varphi_1,\ldots,\varphi_{n-1}><\varphi_0,\varphi_1,\ldots,\varphi_{n-1}| \\ <\widehat{\varphi}|\varphi_0,\varphi_1,\ldots,\varphi_n><\varphi_0,\varphi_1,\ldots,\varphi_n| \end{pmatrix}. \tag{13}$$

Here $|\varphi_0, \varphi_1, \ldots, \varphi_n >$ is called a vector of state in vector Fock space, $\varphi_0, \varphi_1, \ldots, \varphi_n$ are parameters of particles, for example, coordinates, momentum, energy. The secondary quantized function $|\widehat{\varphi} >$ consists of the sum of its projections:

$$< \widehat{\varphi}| = \sum_{n=0}^{\infty} \int \ldots \int d\varphi_0 \ldots d\varphi_n < \widehat{\varphi}|\varphi_0, \varphi_1, \ldots, \varphi_n >< \varphi_0, \varphi_1, \ldots, \varphi_n| \quad (14)$$

where the following identity holds for a vacuum state φ_0:

$$\int d\varphi_0 \equiv 1. \quad (15)$$

Let us assume that the wave function $|\widehat{\varphi} >$ of many particle system is produced by an action of projection operator $\widehat{\varphi}$ on a vector $|\varphi >$:

$$|\widehat{\varphi} >= \widehat{\varphi}|\varphi > . \quad (16)$$

Since the operator $\widehat{\varphi}$ is a projector it possesses a property of self-adjointness. Hence, after taking into account the expression conjugated to (16) in (14) the obtained relationship can be transformed to the following form

$$< \varphi|\, \widehat{\varphi}^{\dagger} = \sum_{n=0}^{\infty} \int \ldots \int d\varphi_0 \ldots d\varphi_n < \varphi|\widehat{\varphi}\varphi_0, \varphi_1, \ldots, \varphi_n >< \varphi_0, \varphi_1, \ldots, \varphi_n|. \quad (17)$$

The multidimensional vector $|\varphi_1, \ldots, \varphi_n >$ belongs to a tensor product of vector spaces $V_1 \otimes V_2 \otimes \ldots \otimes V_n$:

$$|\varphi_0, \varphi_1, \ldots, \varphi_n >= \frac{1}{\sqrt{n!}}|\varphi_0 >\ |\varphi_1 > \ldots |\varphi_n > . \quad (18)$$

Since φ_0 is a vacuum state, the projection $\widehat{\varphi}|\varphi_0 >$ of the vector of vacuum state $|\varphi_0 >$ is also the vacuum state

$$\widehat{\varphi}|\varphi_0 >= |\varphi_0 > . \quad (19)$$

Substituting (19) in (17) and multiplying the obtained expression at the left by a ket - vector $|\varphi >$ we get

$$|\varphi >< \widehat{\varphi}| \equiv |\varphi >< \varphi|\, \widehat{\varphi}^{\dagger}$$

$$= \sum_{n=0}^{\infty} \int \ldots \int d\varphi_0 \ldots d\varphi_n |\varphi >< \varphi|\hat{I}|\varphi_0, \varphi_1, \ldots, \varphi_n >< \varphi_0, \varphi_1, \ldots, \varphi_n|. \quad (20)$$

We see that the n-dimensional vector $|\varphi_0, \varphi_1, \ldots, \varphi_n>$ belongs to a sum of tensor products of vector space V on the $n-1$ -dimensional space $V_1 \otimes V_2 \otimes \ldots \otimes V_{n-1}$:

$$|\varphi_0, \varphi_1, \ldots, \varphi_n> \in V_1 \otimes V_1^{n-1} \oplus V_2 \otimes V_2^{n-1} \oplus \ldots \oplus V_n \otimes V_n^{n-1}, \qquad (21)$$

where the $(n-1)$-dimensional vector spaces V_k^{n-1} are tensor products as

$$V_k^{n-1} = V_0 \otimes V_1 \otimes \ldots \otimes V_{k-1} \otimes V_{k+1} \otimes \ldots \otimes V_n. \qquad (22)$$

Therefore, using (21) and taking into account antisymmetry of the wave function we can rewrite formula (18) as

$$|\varphi_0, \varphi_1, \ldots, \varphi_n> = \frac{1}{\sqrt{n}}(|\varphi_0, \varphi_1> \ |\varphi_2, \ldots, \varphi_n> + (-1)$$
$$\times |\varphi_0, \varphi_2> \ |\varphi_1, \varphi_3, \ldots, \varphi_n> + \ldots + (-1)^{n-1}|\varphi_0, \varphi_n> \ |\varphi_1, \ldots, \varphi_{n-1}>) (23)$$

where the multiplier $(-1)^k$ has arisen because of antisymmetry of the many electron wave function with respect to permutation of particles. Assuming orthonormality of the vectors $|\varphi_i>$: $<\varphi_k|\varphi_i> = \delta(\varphi_k - \varphi_i)$, substituting (23) in the formula (20), and taking into account decomposition (18) we obtain

$$|\varphi><\varphi|\,\widehat{\varphi}^\dagger = \sum_{n=0}^{\infty} \int \ldots \int d\varphi_0 \ldots d\varphi_n \frac{1}{\sqrt{n}} \sum_{i=1}^{\infty} \sum_{k} |\varphi><\varphi|\varphi_i>$$
$$\times (-1)^{k-1}\delta(\varphi_k - \varphi_i)|\varphi_0, \varphi_1, \ldots, \varphi_{k-1}, \varphi_{k+1}, \ldots, \varphi_n><\varphi_0, \varphi_1, \ldots, \varphi_n|. (24)$$

Integrating with account of presence of δ - functions and taking into account identity of particles, we transform the expression (24) to the following form

$$|\varphi><\varphi|\,\widehat{\varphi}^\dagger = \sum_{n=1}^{\infty} \int \ldots \int d\varphi_0 \ldots d\varphi_{n-1}\sqrt{n}\sum_{i=1}^{n}(-1)^{i-1}$$
$$\times |\varphi><\varphi|\hat{I}|\varphi_i> |\varphi_0, \varphi_1, \ldots, \varphi_{i-1}, \varphi_{i+1}, \ldots, \varphi_n><\varphi_0, \varphi_1, \ldots, \varphi_n|. \qquad (25)$$

From expression (25) we find the expansion of the projector $\widehat{\varphi}$:

$$\widehat{\varphi}^\dagger = \sum_{k=1}^{n} |\varphi_k> \widehat{\varphi'_k}^\dagger. \qquad (26)$$

where the operator $\widehat{\varphi'_k}$ is defined by the following expression:

$$\widehat{\varphi'_k}^{\dagger} = (-1)^{k-1} \sum_{n=1}^{\infty} \int d\varphi_0 \dots d\varphi_{n-1} \sqrt{n}$$
$$\times |\varphi_0, \varphi_1, \dots, \varphi_{k-1}, \varphi_{k+1}, \dots, \varphi_n >< \varphi_0, \varphi_1, \dots, \varphi_n|. \tag{27}$$

We construct a basis set of operators on which then we expand the secondary quantized function $< \varphi| \widehat{\varphi}^{\dagger}$. From the expressions (25), (27) and decomposition of operator unity $\hat{I} = \sum_i |\varphi_i >< \varphi_i|$ we find

$$< \varphi| \widehat{\varphi}^{\dagger} = \sum_{i,k=1}^{n} < \varphi|\hat{I}|\varphi_k > \hat{I}|\varphi_i >< \varphi_i|\hat{I}\, \widehat{\varphi'_k}^{\dagger} = \sum_{i,k=1}^{n} a_k(\varphi)|\varphi_i >< \varphi_i|\hat{I}\, \widehat{\varphi'_k}^{\dagger} \tag{28}$$

where factors $a_k(\varphi)$ are defined by the expression $a_k(\varphi) =< \varphi|\hat{I}|\varphi_k >$. Obviously, the constructed secondary quantized wave functions $|\varphi_i >< \varphi_i|\widehat{\varphi'_k}^{\dagger}$, $i, k = 1, 2, \dots$ can be considered as a basis set for the expansion of secondary quantized function $< \varphi| \widehat{\varphi}^{\dagger}$ in a series (28). Taking into account identity of particles we can define the one-particle annihilation operator $\widehat{\varphi'_1}(\varphi_\alpha)$ as [7]

$$\widehat{\varphi'_1}(\varphi_\alpha) \equiv \left.\widehat{\varphi'_{k=1}}^{\dagger}\right|_{\varphi_n \to \varphi_\alpha} = |\varphi_0 >< \varphi_\alpha| + \sqrt{2} \int d\varphi_1 |\varphi_0, \varphi_1 >< \varphi_0, \varphi_1, \varphi_\alpha|$$
$$+ \sqrt{3} \int d\varphi_1 d\varphi_2 |\varphi_0, \varphi_1, \varphi_2 >< \varphi_0, \varphi_1, \varphi_2, \varphi_\alpha| + \dots \tag{29}$$

and express two-particle operators $\widehat{\Phi}^{\dagger}_{pair}$ and $\widehat{\Phi}_{pair}$ (1), (2) describing pairs of particles through these one-particle operators. The secondary quantized wave function (28) contains entangled states and consequently the positronic contribution at the offered way of quantization.

IV. Method of the Green functions

Let us utilize the technique of projection operators for the description of a Green function [8]

$$\hat{G}(z) = \hat{I}^2\, \hat{G}(z) = \int\int d\vec{r} d\vec{r}\,' |\vec{r}\,' >< \vec{r}\,'|\hat{G}(z)|\vec{r} >< \vec{r}|$$
$$= \int\int d\vec{r} d\vec{r}\,' |\vec{r}\,' > G(\vec{r}, \vec{r}\,'; z) < \vec{r}|. \tag{30}$$

Let us define the Green function G for N-dimensional problem as a solution of the following equation:

$$\left[\imath\hbar\frac{\partial}{\partial t} - \hat{H}_0 - \hat{H}_N(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)\right] G(\vec{r}_1, \vec{r}\,'_1; \vec{r}_2, \vec{r}\,'_2; \ldots; \vec{r}_N, \vec{r}\,'_N; t, t')$$
$$= \delta(\vec{r}_1 - \vec{r}\,'_1) \ldots \delta(\vec{r}_N - \vec{r}\,'_N)\delta(t - t'). \quad (31)$$

Here $\hat{H}_0$ is the kinetic energy of particles, $\hat{H}_N(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$ is an interaction operator determined as

$$\hat{H}_N(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N) = \sum_{i<j}^{N} \hat{H}_1(|\,\vec{r}_i - \vec{r}_j\,|). \quad (32)$$

We can describe the Green function of N-dimensional problem as

$$\begin{aligned} \hat{G}_N(t_1 - t_0) &= \hat{G}_N^{(0)}(t_1 - t_0) \\ &+ \int \hat{G}_N^{(0)}(t_1 - t_i)\, \hat{H}_N(\vec{r}_1(t_i), \vec{r}_2(t_i), \ldots, \vec{r}_N(t_i))\, \hat{G}_N^{(0)}(t_i - t_0) dt_i + \int dt_i \\ &\times \int dt_k\, \hat{G}_N^{(0)}(t_1 - t_i)\, \hat{H}_N(\vec{r}_1(t_i), \vec{r}_2(t_i), \ldots, \vec{r}_N(t_i))\, \hat{G}_N^{(0)}(t_i - t_k) \\ &\times \hat{H}_N(\vec{r}_1(t_k), \vec{r}_2(t_k), \ldots, \vec{r}_N(t_k))\, \hat{G}_N^{(0)}(t_k - t_0) + \ldots = \hat{G}_N^{(0)}(t_1 - t_0) \\ &+ \int dt_i\, \hat{G}_N^{(0)}(t_1 - t_i)\, \hat{H}_N(\vec{r}_1(t_i), \vec{r}_2(t_i), \ldots, \vec{r}_N(t_i)) \Big[\hat{G}_N^{(0)}(t_i - t_0) \\ &+ \int dt_k\, \hat{G}_N^{(0)}(t_i - t_k)\, \hat{H}_N(\vec{r}_1(t_k), \vec{r}_2(t_k), \ldots, \vec{r}_N(t_k))\, \hat{G}_N^{(0)}(t_k - t_0) + \ldots\Big] \\ &= \hat{G}_N^{(0)}(t_1 - t_0) \\ &+ \int dt_i\, \hat{G}_N^{(0)}(t_1 - t_i)\, \hat{H}_N(\vec{r}_1(t_i), \vec{r}_2(t_i), \ldots, \vec{r}_N(t_i))\, \hat{G}_N(t_i - t_0) \quad (33) \end{aligned}$$

where the projection N-particle Green function is defined by the following expression:

$$\begin{aligned} \hat{G}_N(t) &= \int d\vec{r}_1 \vec{r}_2 \ldots \vec{r}_N d\vec{r}\,'_1 d\vec{r}\,'_2 \ldots d\vec{r}\,'_N \\ &\times |\vec{r}\,'_1, \vec{r}\,'_2, \ldots, \vec{r}\,'_N >< \vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N| \\ &\times \hat{G}(t) |\vec{r}\,'_1, \vec{r}\,'_2, \ldots, \vec{r}\,'_N >< \vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N| \equiv \int d\vec{r}_1 \ldots \vec{r}_N d\vec{r}\,'_1 \ldots d\vec{r}\,'_N \\ &|\vec{r}\,'_1, \ldots, \vec{r}\,'_N >< G(\vec{r}_1, \ldots, \vec{r}_N, \vec{r}\,'_1, \ldots, \vec{r}\,'_N; t) >< \vec{r}_1, \ldots, \vec{r}_N| \quad (34) \end{aligned}$$

and the interaction is determined by a projector

$$\begin{aligned} \hat{H}_N &= \hat{I}^2\, \hat{H}_1 = \int d\,\vec{r}_1 \ldots \vec{r}_N\, d\vec{r}\,'_1 \ldots d\vec{r}\,'_N \\ &\times |\vec{r}\,'_1, \ldots, \vec{r}\,'_N >< \vec{r}\,'_1, \ldots, \vec{r}\,'_N\,|\,\hat{H}_1\,|\,\vec{r}_1, \ldots, \vec{r}_N >< \vec{r}_1, \ldots, \vec{r}_N\,|. \quad (35) \end{aligned}$$

In the secondary quantized case the operators $\hat{G}_N$, $\hat{H}_N$ become products of projection operators:

$$\hat{G}_N(t'_1 - t_1, \ldots, t'_N - t_N)$$
$$= \int d\vec{r}_1 \ldots \vec{r}_N \, d\vec{r}\,'_1 \ldots d\vec{r}\,'_N \, \varphi^+(\vec{r}\,'_1, t_1) \ldots \varphi^+(\vec{r}\,'_N, t_N)|0>$$
$$\times G(\vec{r}_1, t_1; \ldots, \vec{r}_N, t_N; \vec{r}\,'_1, t_1; \ldots, \vec{r}\,'_N, t_N) < 0|\varphi^-(\vec{r}_1, t_1) \ldots \varphi^-(\vec{r}_N, t_N), \quad (36)$$

$$\hat{H}_N = \frac{1}{2}\sum_{i,j} \int d\vec{r}_i \, \vec{r}_j \, d\vec{r}\,'_i \; d\vec{r}\,'_j \, \varphi^+(\vec{r}\,'_i)\varphi^+(\vec{r}\,'_j)|0>$$
$$\times < \vec{r}\,'_i, \vec{r}\,'_j \,|\, \hat{H}_1 \,|\, \vec{r}_i, \vec{r}_j > \delta(\vec{r}_i - \vec{r}\,'_i)\delta(\vec{r}_j - \vec{r}\,'_j) < 0|\varphi^-(\vec{r}_i)\varphi^-(\vec{r}_j)$$
$$= \frac{1}{2}\sum_{i,j} \int d\vec{r}_i \, \vec{r}_j \, dt_j \varphi^+(\vec{r}_i, t_i)\varphi^+(\vec{r}_j, t_j)|0>$$
$$\times \hat{H}_1(|\,\vec{r}_i - \vec{r}_j\,|)\delta(t_i - t_j) < 0|\varphi^-(\vec{r}_i, t_i)\varphi^-(\vec{r}_j, t_j) \quad (37)$$

where the time $t_i(t'_i)$, $i = 1, \ldots, N$ is defined as $t_i = t + \varepsilon_i(t'_i = t' + \varepsilon'_i)$, $\varepsilon_i(\varepsilon'_i) \to 0$ and in this sense the equality of the secondary quantized wave functions (28) $\varphi^+(\vec{r}_i, t_i) = \varphi^+(\vec{r}_i(t))$ $(\varphi^-(\vec{r}_i, t_i) = \varphi^-(\vec{r}_i(t)))$ is understood. Knowing unperturbed Green function, the perturbed two-particle operator Green function can be found from the equation

$$(\hat{\hat{G}}_2)^{ns'ms} \stackrel{def}{=\!=} \varphi^+(n)\varphi^+(s')\varphi^-(m)\varphi^-(s) = (\varphi^{(0)})^+(n)(\varphi^{(0)})^+(s')$$
$$\times(\varphi^{(0)})^-(m)(\varphi^{(0)})^-(s) + \frac{1}{2}\int dt_i dt_j d\vec{r}_i \, \vec{r}_j \, d\vec{r}\,'_i \; d\vec{r}\,'_j$$
$$\times\delta(\vec{r}_i - \vec{r}\,'_i)\delta(\vec{r}_j - \vec{r}\,'_j)\Big[(\varphi^{(0)})^+(j')(\varphi^{(0)})^+(s')(\varphi^{(0)})^-(m)(\varphi^{(0)})^-(i')$$
$$\times\varphi^+(\vec{r}\,'_j, t'_j)\varphi^+(\vec{r}\,'_i, t'_i)\,\hat{H}_1(|\,\vec{r}_i - \vec{r}_j\,|)\delta(t_i - t_j)\varphi^-(\vec{r}_i, t_i)\varphi^-(\vec{r}_j, t_j)$$
$$\times\varphi^+(n)\varphi^+(i)\varphi^-(j)\varphi^-(s) + (\varphi^{(0)})^+(j')(\varphi^{(0)})^+(s')(\varphi^{(0)})^-(m)(\varphi^{(0)})^-(i')$$
$$\times\varphi^+(\vec{r}\,'_i, t'_i)\varphi^+(\vec{r}\,'_j, t'_j)\,\hat{H}_1(|\,\vec{r}_i - \vec{r}_j\,|)\delta(t_i - t_j)\varphi^-(\vec{r}_j, t_j)\varphi^-(\vec{r}_i, t_i)$$
$$\times\varphi^+(n)\varphi^+(i)\varphi^-(j)\varphi^-(s)\Big] \, . (38)$$

V. Discussion and conclusion

We observe that the contributions from particles and particle-antiparticle pairs are not represented by superposition due to anticommutators that are not equal to zero. In this sense these contributions are nonseparable.

For one-particle Dirac problem the secondary quantized wave function is expressed only through one-particle creation operators for particles and antiparticles. In this case one can neglect the nonseparability of contributions from particles and particle-antiparticle pairs. Therefore instead of operator two-particle Green function $(\widehat{\hat{G}}_2)^{ns'ms}$ one can introduce an operator Green function $(\hat{G}_2)^{ms'sn}$ obtained from it by even permutations of the creation and annihilation operators:

$$(\hat{G}_2)^{ms'sn} = \varphi^-(m)\varphi^+(s')\varphi^-(s)\varphi^+(n) \quad \text{for } t_m > t_{s'}, t_s > t_n. \tag{39}$$

By rewriting eq. (38) for the Green function (39) and summation over indexes $s,\ s'$ it is possible to obtain a Dyson equation describing one-particle Green function. Finally, we have shown that in the case of variable number of electron and fermionic pairs it is necessary to utilize more general equation for the Green functions which allows to describe any combination of particles and pairs.

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