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ON OPTIMIZATION OF THE FUNCTIONAL RENORMALIZATION GROUP APPLIED TO THE QUANTIZED ANHARMONIC OSCILLATOR

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Abstract

The quantized anharmonic oscillator is studied by the functional renormalization group method. It is shown that the energy gap between the first excited state and the ground state exhibits a strong scheme dependence. We argue that there is no optimal value of the regulator parameter that gives the closest results to the exact energy gaps for any values for the initial couplings.

I. Introduction

In this work we apply the functional renormalization group (RG) method for the quantized anharmonic oscillator. Quantum mechanics can be considered as a quantum field theory, with 0 spatial and 1 time dimension. In the framework of the path integral formalism we can use the functional renormalization group method in quantum mechanics as in quantum field theory. The RG method is widely used in many areas of modern physics [1, 2, 3, 4, 5, 6, 7, 8, 9, 10], starting from the theory of phase-transitions in condensed matter systems to some aspects of cosmological problems. Although quantum mechanics can be investigated by solving the Schrödinger

equation numerically the RG method could give a new point of view of the problem.

We investigate the quantized anharmonic oscillator in one space dimension by solving the RG equations [7, 11]. In the case of this simple model it is possible to compare our results with other method's results, where the problem is treated by solving the Schrödinger equation numerically with the corresponding potential, which are considered as the exact values henceforward. It provides us a good testing ground to investigate how we can optimize the RG method. We compute the energy gap between the first excited state and the ground state with the RG technique and compare it with the exact results. The determination of the energy gap is an interesting question in itself because it leads a non-perturbative problem when the potential has two degenerate minima [10].

II. Evolution equations

The RG method provides us a partial integro-differential equation for the effective action, which is called the Wetterich equation [1, 7]

$$\dot{\Gamma}_k = \frac{1}{2}\mathrm{Tr}\frac{\dot{R}_k}{R_k + \Gamma_k''}, \tag{1}$$

where $\dot{} = k\partial_k$, $' = \partial/\partial\phi$, R_k is the regulator and the trace Tr denotes the integration over all momenta and summation for internal indices. Eq. (1) has been solved over the functional subspace defined by the ansatz

$$\Gamma_k = \int_x \left[\frac{Z_k}{2}(\partial_\mu\phi)^2 + V_k\right], \tag{2}$$

with the potential V_k, and the wave function renormalization Z_k. In case of the local potential approximation (LPA) $Z_k = 1$. Then the evolution equation for the potential reads as

$$\dot{V}_k = \frac{1}{2\pi}\int_0^\infty dp \frac{\dot{R}_k}{p^2 + R_k + V_k''}. \tag{3}$$

in one space dimension. This equation can be considered as the RG evolution equation of quantum mechanics. The initial condition for the Wetterich

equation is given by the explicit form of the effective action at the ultra violet (UV) cutoff $k = \Lambda$, i.e., by that of the classical action of the anharmonic oscillator by fixing the initial values of the couplings. There are lots of examples in the literature for different type of regulator functions. Here we use the following power-law type regulator function

$$R_k(p) = p^2 \left(\frac{k^2}{p^2}\right)^b, \tag{4}$$

where $b \geq 1$. The particular cases $b \to \infty$ and $b = 1$ correspond to the Wegner-Houghton (WH) and the Callan-Symanzik (CS) renormalization schemes, respectively.

We investigate the quantum mechanical anharmonic oscillator. We apply a polynomial ansatz

$$V_k = \frac{m_k^2}{2}\phi^2 + g_k\phi^4 + \sum_{n=3} \frac{g_{2n}(k)}{(2n)!}\phi^{2n} \tag{5}$$

for the potential, where we introduced the additional couplings g_{2n} with $n \geq 3$ which are vanishing at the UV scale. Applying Eq. (5) in Eq. (3) we obtain a system of ordinary differential equations for the couplings. We notice that in the cases of the WH and CS schemes in LPA the integral with respect to the momentum p has a closed form, so that the equations become more simple.

III. Results

The quantized anharmonic oscillator has been investigated in a traditional way by the solution of the Schrödinger equation numerically, so we can control our RG results for the energy gap. There are known results in the literature where the problem was treated in the framework of the Wetterich equation [12] or in that of the heat-kernel technique [13]. Here we consider a broader set of regulator functions. The UV potential has the form of

$$V_\Lambda = \frac{m_\Lambda^2}{2}\phi^2 + g_\Lambda\phi^4, \tag{6}$$

where m_Λ^2 and g_Λ denotes the initial values of the couplings, and the further couplings are suppressed. We investigate the energy gap as the function of the initial values. Mathematically they provide different initial conditions of RG equations which constitute a system of ordinary first-order, but highly non-linear differential equations.

We have developed a C++ code in order to solve the system of the differential equations and applied a 4th order Runge-Kutta method. We chose $\Lambda = 1500$ for the UV cut-off.

Our goal is to determine the effective potential V_0, i.e. the potential in the $k \to 0$ limit. We note that the physical properties of the model can be determined from the knowledge of the effective potential. In LPA the value of the energy gap is the root of the second derivate of the effective potential at the vacuum expectation value $< \phi >$ of the field variable [10],

$$\Delta E = \sqrt{\frac{\partial^2 V_0}{\partial \phi^2}}\Bigg|_{\phi=<\phi>}. \tag{7}$$

In quantum mechanics the vacuum expectation value $< \phi >$ is generally the trivial field configuration $< \phi >= 0$, i.e. we have to take the second derivative of the effective potential at $\phi = 0$. In our case the energy gap is

$$\Delta E = m_0, \tag{8}$$

which is the IR limit of the coupling m_k. Figure 1 shows the flow of the coupling m_k during the evolution in WH scheme for various initial values for the potential. In the IR limit the values of the dimensionful couplings scale marginally, i.e. they tend to positive constant values.

We chose the power-law regulator, where the parameter b is freely chosen. By varying b one can obtain different values of the energy gap ΔE. The results also depend on the initial values of the couplings and on the number N of the couplings taken into account in the potential. According to previous results in the literature it was shown that in the 2-dimensional sine-Gordon model [14, 15, 16], or in the 3-dimensional and O(N) model [13, 17] the optimal value of b is around 2, which means that $b \approx 2$ provides us the closest value of the RG results to the exact ones. Our aim is to find the optimal value of b, since it was not investigated so far in quantum mechanical models.

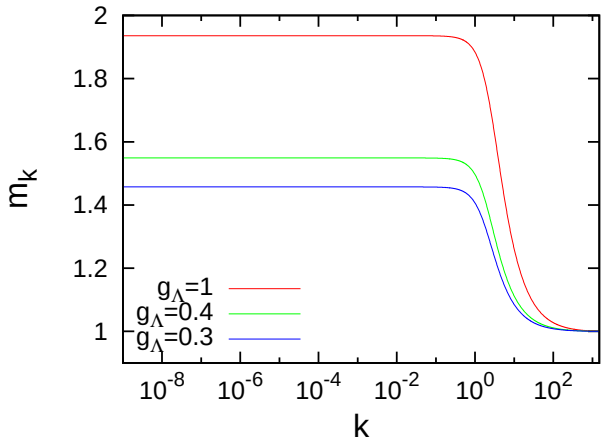

Figure 1: The evolution of the couplings m_k is shown. The curves correspond to different initial values of g_Λ, we chose $m_\Lambda^2 = 1$.

First we investigated the N dependence of the energy gap. Our results are demonstrated in figure 2. We choose the the case $m_\Lambda^2 < 0$ for the optimization, because then the deviation from the exact value is larger. We note that although larger values of N could improve the approximation of the expansion but it causes larger numerical errors. Due to the numerical approximation there is an optimal value of the number of couplings, which is about 6 in the local potential approximation.

One can distinguish two phases of the classical model depending on the sign of m_k^2, because the potential can have either a single minimum at the origin or 2 minima for non-trivial values of the field variable. The second case is considered as the spontaneously broken symmetric phase, where the ground state breaks the Z_2 symmetry of the model. In quantum mechanics (i.e., in $0+1$-dimensional quantum field theory) the effective potential is symmetric due to the tunneling effect even if $m_\Lambda^2 < 0$, so the model has a single symmetric phase. However our numerical results show that for negative values of m_Λ^2 and for large values of g_Λ the effective potential becomes concave at $\phi = 0$. This is due to the strong truncation of the potential and that of the gradient expansion.

As to the next, we investigated the optimization with respect to the

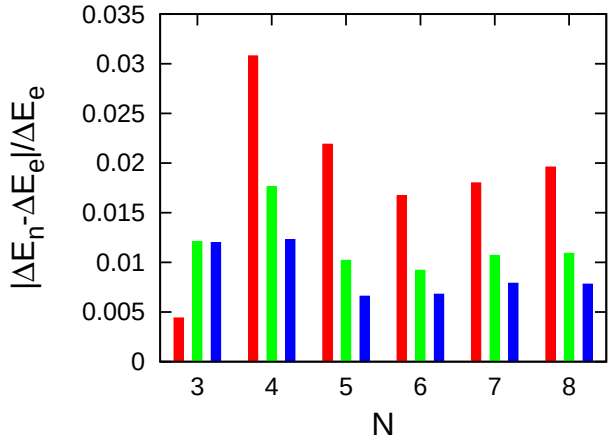

Figure 2: The difference between the exact and numerical values of the energy gap as the function of N is shown for various initial values of g_Λ and m^2_Λ. The red column corresponds to $g_\Lambda = 0.4$, the green column denotes $g_\Lambda = 0.3$ and blue column refers to $g_\Lambda = 0.2$. The data were calculated in the CS scheme with $m^2_\Lambda = -1$.

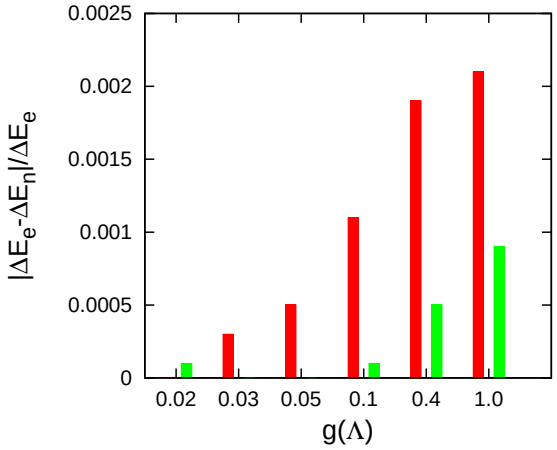

Figure 3: The relative error of the energy gap as the function of g_Λ is shown in the case of $m^2_\Lambda = 1$ for the number of couplings $N = 6$. The red column denotes the WH scheme, the green column denotes the CS scheme.

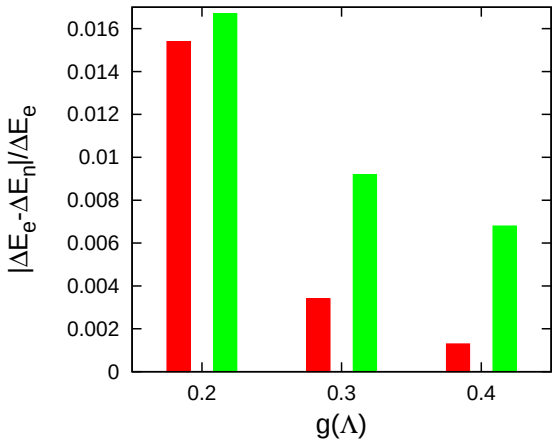

Figure 4: The relative error of the energy gap as the function of g_Λ for $m^2_\Lambda = -1$ and the number $N = 6$ of the couplings. The red column denotes the WH scheme, the green column denotes the CS scheme.

renormalization scheme. We are free to choose the regulator function, and naturally the results depend on the renormalization scheme due to the applied approximations, the LPA and the Taylor expansion. Figures 3 and 4 show the comparison of the results of the WH and CS schemes for various initial values, in the case of six couplings. In our approximation for $m^2_\Lambda > 0$ the CS scheme gives better results than the WH scheme. Our results are in good agreement with the exact values when the parameter g is tiny and the convergence of the expansion is faster. For $m^2_\Lambda < 0$ the statement is just the opposite, the WH scheme works better and our results are the more bad the smaller is g_Λ, i.e., the more we approach the regime where the two minima of the bare potential become rather flat.

We investigated the optimization with respect to the parameter b for various numbers of couplings and various their various initial values. We chose $m^2_\Lambda = 1, -1$ and $g_\Lambda = 1, 0.4, 0.1, 0.05, 0.03, 0.02, -0.4, 0.3, -0.2, -0.1$. Generally we can evaluate the integral by numerically for arbitrary values of b. We have applied the Romberg-method in our code [18]. The parameter b dependence is shown in Figure 5.

Although the single case shown in Figure 5 would indicate that the op-

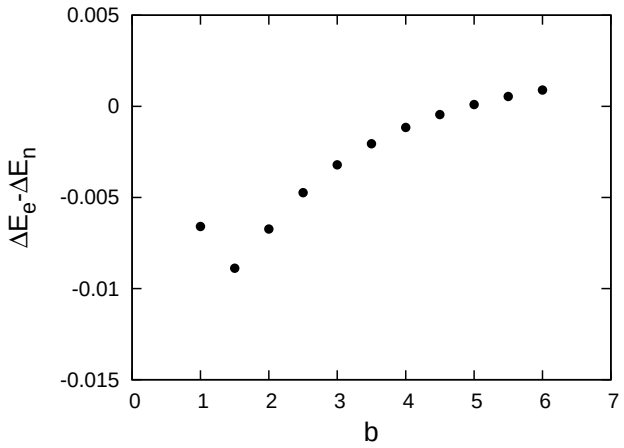

Figure 5: The deviation of ΔE as the function of the parameter b is shown for $m_\Lambda^2 = -1$, $g_\Lambda = 0.4$, and $N = 6$.

timal choice were $b \approx 1.5$, this is not the case when we consider the cases with various numbers of couplings and their various initial values. According to our findings the optimal value of b depends rather remarkably on the number of couplings and on their initial values. These optimal values range from $b = 2.5$ to $b = 5.5$. Generally, the optimal value is less than 4.5 for $N = 4$ and it is between 4.5 and 5.5 for $N = 6$, for the initial values of the couplings investigated by us.

IV. Conclusions

By using the functional renormalization group method we calculated the energy gap for the quantized anharmonic oscillator. The renormalization requires approximations to be introduced. We used the local potential approximation and the Taylor expansion of the potential with various truncations. The energy gap has been determined for different regulator functions. Our results have been compared with the exact numerical results taken from the literature that were obtained numerically by solving the Schrödinger equation for the anharmonic oscillator. We showed that one cannot find a unique regulator which can give the closest results to the exact ones for all values of them bare couplings. It suggests that we should

perform the optimization for the regulator for every model, since there is no optimized regulator that is suitable in every calculation.

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