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MOMENTUM DEPENDENT WAVE-FUNCTION RENORMALIZATION FOR THE SINE-GORDON MODEL

J. Kovács[1], S. Nagy[1], K. Sailer[1]

[1] Department of Theoretical Physics, University of Debrecen, P.O. Box 5, H-4010 Debrecen, Hungary

Abstract

The functional renormalization group equations are presented in the two-dimensional sine-Gordon model for momentum dependent wave-function renormalization in the gradient expansion. The flow equations are derived for the compactly supported smooth infrared regulator function.

I. Introduction

The sine-Gordon (SG) model [1] has been investigated in framework of functional renormalization group (RG) [3]. Calculations with momentum independent wavefunction renormalization seem sufficient to describe the model qualitatively. By our expectations neglecting momentum dependence does not change significantly the low energy behavior of the model. However there are some questions (scheme dependence or asymptotic safety) which merit further investigations and should be answered by calculating further term of wave-function renormalization. Furthermore it is also an open issue to investigate whether the gradient expansion with the new term is convergent or not and whether it can modify the phase structure of the SG model or leave it qualitatively unchanged. In this article equations with the momentum dependent term are derived in the case when the wavefunction renormalization does not depend from the field variable. The higher-order equations can lead to more precise investigation of the model.

The paper is organized as follows. In section II. we briefly discuss the SG model and in section III. we derive the RG equations and give the explicit form of the linearized RG equations in the case of Callan-Symanzik scheme. In the summary the conclusions are drawn up.

II. The model

The two-dimensional (2D) SG model is a self-interacting scalar model defined by the bare action

$$S = \int_x \left[\frac{1}{2} (\partial_\mu \phi)^2 + u \cos(\beta \phi) \right], \tag{1}$$

in Euclidean spacetime. The SG model belong to the university class of the 2D Coulomb gas and the 2D-XY spin model, hence it has important applications in condensed matter systems [2], since the model exhibits a Kosterlitz-Thouless (KT) type or infinite order phase transition. There are high energy applications of SG model, too. In spite of its simplicity it has nontrivial phase structure which feature is common with the non-Abelian gauge theories. It can be applied as a toy model to investigate new methods for example soft mechanism of quark confinement [4] or string theory [5].

The periodicity of the potential the model plays an important role of understanding its low energy behavior. Describing the model with the usual perturbative treatment cannot be applied, because truncating the Taylor series expansion in the field variable breaks the periodic symmetry. Functional RG method preserves the symmetry and it has been successfully applied to investigate the phase structure of the model. The method is capable of giving the KT type phase transition and the corresponding critical exponents.

III. Renormalization

The RG method provides us the following functional equation for the effective action Γ, which is called Wetterich-equation [3]

$$k\partial_k \Gamma_k = \frac{1}{2} \mathrm{Tr} \frac{k\partial_k R_k}{R_k + \Gamma_k''}, \tag{2}$$

where $' = \frac{\partial}{\partial\phi}$, R_k is the regulator and the trace Tr denotes integration over all momenta and summation for internal indices. The mass like term $R_k[\phi]$ is defined by the regulator function $R_k(p^2)$

$$R_k[\phi] = \frac{1}{2}\phi \cdot R_k(p^2) \cdot \phi, \tag{3}$$

where dots denote integrations. There are several possibilities to choose the regulator function, the following examples are often used in the literature:

$$\begin{aligned} R_{\text{powerlaw}} &= p^2 \left(\frac{1}{y}\right)^b \text{ where } \ b \geq 1, \\ R_{\text{Litim}} &= \left(k^2 - p^2\right) \theta \left(k^2 - p^2\right), \\ R_{\text{exponential}} &= \frac{p^2}{e^y - 1}, \end{aligned} \tag{4}$$

where

$$y = \frac{p^2}{k^2}. \tag{5}$$

We chose the so-called compactly supported smooth (css) regulator function [6, 7]

$$R_{\text{css}} = \frac{s_1 p^2}{\exp[s_1 y^b/(1 - s_2 y^b)] - 1} \theta \left(1 - s_2 y^b\right), \tag{6}$$

where s_1, s_2 are positive parameters, $b \geq 1$ and θ is the Heaviside step function. The css regulator reproducts the regulator functions in (4) as special cases:

$$\begin{aligned} \lim_{s_1 \to 0, s_2 \to 0} R_{\text{css}} &= R_{\text{powerlaw}}, \\ \lim_{s_1 \to 0, s_2 \to 1} R_{\text{css}} &= R_{\text{Litim}} \text{ if } \ b = 1, \\ \lim_{s_1 \to 1, s_2 \to 0} R_{\text{css}} &= R_{\text{exponential}} \text{ if } \ b = 1. \end{aligned} \tag{7}$$

This important property allows us to compare the results given by different regulators [7]. In this paper we choose $b = 1$ in order to satisfy the normalization conditions

$$\begin{aligned} \lim_{y \to 0} \frac{R_k(p)}{k^2} &= 1, \\ \lim_{y \to \infty} \frac{R_k(p)}{k^2} &= 0. \end{aligned} \tag{8}$$

During the calculations we use several approximations to derive differential equations from the original functional equation. According to the gradient expansion we can expand the action in powers of the derivatives of the field

$$\begin{aligned}\Gamma_k[\phi] &= \int_x \Big[V_k(\phi) + \frac{1}{2}Z_k(\phi)(\partial_\mu\phi)^2 \\ &\quad + H_1(\phi)(\partial_\mu\phi)^4 + H_2(\phi)(\Box\phi)^2 + \dots\Big], \end{aligned} \tag{9}$$

where $V_k(\phi)$ is the potential and $Z_k(\phi)$ is the wave-function renormalization. We choose

$$V = u\cos(\beta\phi) \tag{10}$$

for the potential. In Fourier space we can write the field independent wave-function renormalization as

$$Z_k = Z_k(P^2) = z + z_1 P^2 + \mathcal{O}(P^4). \tag{11}$$

Neglecting z_1 we can derive the following equations

$$k\partial_k V_k = \frac{1}{2}\int_p \frac{k\partial_k R_{k,p}}{p^2 Z_k(p^2) + R_{k,p} + V_k''}, \tag{12}$$

$$\begin{aligned}P^2\partial_k Z_k(P^2) &= \mathcal{P}_0 \int_p \frac{k\partial_k R_{k,p}(V_k''')^2}{(p^2 Z_k(p^2) + R_{k,p} + V_k'')^2} \times \\ &\quad \Bigg(\frac{1}{(P+p)^2 Z_k((P+p)^2) + R_{k,P+p} + V_k''} \\ &\quad - \frac{1}{[p^2 Z_k(p^2) + R_{k,p} + V_k'']}\Bigg), \end{aligned} \tag{13}$$

where the projection operator

$$\mathcal{P}_0 = \frac{1}{2\pi}\int_0^{2\pi} d\phi \tag{14}$$

provides the first Fourier mode. By the expansion in P we can derive the RG equations for the couplings z and z_1. We introduce the following new

variables:

$$
\begin{aligned}
C_0 &= 2z + 4z_1 p^2, \\
C_1 &= \frac{2}{k^2}, \\
K &= \frac{s_1 y}{1 - s_2 y}, \\
C_2 &= e^K \frac{C_1 s_1}{(1 - s_2 y)^2}, \\
C_3 &= e^K \frac{C_1^2 s_1}{(1 - s_2 y)^3} \left(\frac{s_1}{1 - s_2 y} + 2 s_2 \right), \\
C_4 &= e^K \frac{C_1^3 s_1}{(1 - s_2 y)^4} \left(\frac{s_1^2}{(1 - s_2 y)^2} + \frac{6 s_2 s_1}{(1 - s_2 y)} + 6 s_2^2 \right), \\
C_5 &= e^K \frac{C_1^4 s_1}{(1 - s_2 y)^5} \left(\frac{s_1^3}{(1 - s_2 y)^3} + \frac{12 s_1^2 s_2}{(1 - s_2 y)^2} + \frac{36 s_2^2 s_1}{(1 - s_2 y)} + 24 s_2^3 \right), \\
N &= e^K - 1, \\
C_6 &= 2N^{-3} C_2^2 - N^{-2} C_3, \\
C_7 &= -N^{-2} C_2, \\
C_8 &= -6N^{-4} C_2^3 + 6N^{-3} C_3 C_2 - N^{-2} C_4, \\
C_9 &= 24N^{-5} C_2^4 - 36N^{-4} C_3 C_2^2 + 8N^{-3} C_4 C_2 + 6N^{-3} C_3^2 - N^{-2} C_5, \\
C_{10} &= C_0 + s_1 k^2 \theta (1 - s_2 y) \left(\frac{2}{k^2} N^{-1} - y N^{-2} C_2 \right), \\
C_{11} &= 8 z_1 + s_1 k^2 \theta (1 - s_2 y) \left(-\frac{4}{k^2} N^{-2} C_2 + y C_6 \right), \\
C_{12} &= s_1 k^2 \theta (1 - s_2 y) \left(\frac{6}{k^2} C_6 + y C_8 \right), \\
C_{13} &= s_1 k^2 \theta (1 - s_2 y) \left(\frac{8}{k^2} C_8 + y C_9 \right), \\
\mathcal{D}_0 &= z p^2 + z_1 p^4 + R_{k,p} + V_k'', \\
C_{14} &= \frac{2 C_{10}^2}{\mathcal{D}_0^3} - \frac{C_{11}}{\mathcal{D}_0^2},
\end{aligned}
$$

$$
\begin{aligned}
C_{15} &= -\frac{C_{10}}{\mathcal{D}_0^2}, \\
C_{16} &= 24\frac{C_{10}^4}{\mathcal{D}_0^5} - 36\frac{C_{11}C_{10}^2}{\mathcal{D}_0^4} + 8\frac{C_{12}C_{10}}{\mathcal{D}_0^3} + 6\frac{C_{11}^2}{\mathcal{D}_0^3} - \frac{C_{13}}{\mathcal{D}_0^2}, \\
C_{17} &= -6\frac{C_{10}^3}{\mathcal{D}_0^4} + 6\frac{C_{11}C_{10}}{\mathcal{D}_0^3} - \frac{C_{12}}{\mathcal{D}_0^2}, \\
C_{18} &= 2\frac{C_{10}^2}{\mathcal{D}_0^3} - \frac{C_{11}}{\mathcal{D}_0^2}, \\
C_{19} &= \frac{k\partial_k R_{k,p}(V_k''')^2}{(p^2(z+p^2z_1)+R_{k,p}+V_k'')^2}.
\end{aligned} \tag{15}
$$

By these variables the RG equations become

$$
\begin{aligned}
\partial_k z &= \mathcal{P}_0 \int_p \frac{C_{19}}{2}\left(C_{14}\frac{p^2}{2} + C_{15}\right), \\
\partial_k z_1 &= \mathcal{P}_0 \int_p \frac{C_{19}}{8}\left(C_{16}\frac{p^4}{8} + C_{17}p^2 + C_{18}\right).
\end{aligned} \tag{16}
$$

In general the momentum integral cannot be performed analytically. For simplicity we henceforward consider the case of Callan-Symanzik scheme when the regulator function is $R_k = k^2$, it does not depend on p

$$
k\partial_k V_k = \int_p \frac{k^2}{z_1p^4 + zp^2 + k^2 + V_k''}, \tag{17}
$$

$$
\begin{aligned}
k\partial_k z &= \mathcal{P}_0 \int_p k^2 \left(V_k'''\right)^2 \left(-\frac{C_0 + 4z_1p^2}{\mathcal{D}_0^4} + \frac{p^2C_0^2}{\mathcal{D}_0^5}\right), \\
k\partial_k z_1 &= \mathcal{P}_0 \int_p \frac{k^2\left(V_k'''\right)^2}{12}\Bigg(-\frac{8z_1}{\mathcal{D}_0^4} + \frac{48z_1C_0p^2 + 2C_0^2 + 48z_1^2p^4}{\mathcal{D}_0^5} \\
&\quad -\frac{6C_0^3p^2 + 32z_1C_0^2p^4}{\mathcal{D}_0^6} + \frac{3C_0^4p^4}{\mathcal{D}_0^7}\Bigg).
\end{aligned} \tag{18}
$$

Performing the projection we obtain

$$
\begin{aligned}
k\partial_k z &= \frac{k^2u^2}{2}\int_p \frac{-(C_0+4z_1p^2)a_1}{(a_1^2-u^2)^{\frac{5}{2}}} + \frac{p^2C_0^2(4a_1^2+u^2)}{4(a_1^2-u^2)^{\frac{7}{2}}}, \\
k\partial_k z_1 &= \int_p \frac{k^2u^2}{24}\Bigg(\frac{a_2a_1}{(a_1^2-u^2)^{\frac{5}{2}}} + \frac{a_3(4a_1^2+u^2)}{4(a_1^2-u^2)^{\frac{7}{2}}} \\
&\quad + \frac{a_4a_1\sqrt{\frac{a_1+u}{a_1-u}}(4a_1^2+3u^2)}{4(a_1-u)^4(a_1+u)^5} + \frac{a_5(8a_1^4+12a_1^2u^2+u^4)}{8(a_1^2-u^2)^{\frac{11}{2}}}\Bigg), \quad (19)
\end{aligned}
$$

where

$$
\begin{aligned}
a_1 &= zp^2 + z_1p^4 + k^2, \\
a_2 &= -8z_1, \\
a_3 &= 48z_1C_0p^2 + 2C_0^2 + 48z_1^2p^4, \\
a_4 &= -6C_0^3p^2 - 32z_1C_0^2p^4, \\
a_5 &= 3C_0^4p^4. \quad (20)
\end{aligned}
$$

The projection

$$
\mathcal{P}_1 = \frac{1}{2\pi}\int_0^{2\pi} d\phi \cos\phi. \quad (21)
$$

for the the dimensionless coupling $\tilde{u}$ gives

$$
k\partial_k\tilde{u} = -2\tilde{u} + \frac{1}{\tilde{u}}\int_{\tilde{p}} 2\left(\frac{\tilde{a}_1}{\sqrt{(\tilde{a}_1)^2-\tilde{u}^2}} - 1\right), \quad (22)
$$

where tildes denote dimensionless quantities, e.g. $\tilde{u} = u/k^2$. The leading order evolution equations in $\tilde{u}$ are

$$
k\partial_k\tilde{u} = \begin{cases} \frac{\tilde{u}}{4\pi}\frac{1}{\tilde{z}_1^2}\frac{\tilde{x}_2^2-\tilde{x}_1^2+2\tilde{x}_1\tilde{x}_2 ln\left(\frac{\tilde{x}_1}{\tilde{x}_2}\right)}{\tilde{x}_1\tilde{x}_2(\tilde{x}_1-\tilde{x}_2)^3} - 2\tilde{u} & \text{if } D<0, \\ \frac{\tilde{u}}{4\pi}\frac{1}{\tilde{z}_1^2}\left(-\frac{1}{3\tilde{x}_1^3}\right) - 2\tilde{u} & \text{if } D=0, \\ \frac{\tilde{u}}{4\pi}\left(\frac{4\tilde{z}_1}{D\sqrt{D}}\left(\frac{\pi}{2} - tan^{-1}(\frac{\tilde{z}}{\sqrt{D}})\right) - \frac{\tilde{z}}{D}\right) - 2\tilde{u} & \text{if } D>0, \end{cases} \quad (23)
$$

where

$$
\begin{aligned}
\tilde{z} &= z, \\
\tilde{z}_1 &= z_1 k^2, \\
D &= 4\tilde{z}_1 - \tilde{z}^2, \\
\tilde{x}_1 &= \frac{-\tilde{z} + \sqrt{-D}}{2\tilde{z}_1}, \\
\tilde{x}_2 &= \frac{-\tilde{z} - \sqrt{-D}}{2\tilde{z}_1}.
\end{aligned} \tag{24}
$$

As is known [1] if we neglect the scaling of z_1 then the limit $\tilde{u} \to 0$ is determined by z, i.e. $\tilde{u}$ scales in relevant manner for $1/z < 8\pi$ and in irrelevant manner for $1/z > 8\pi$. Coupling z_1 scales in irrelevant manner according to its mass dimension and if z_1 is sufficiently large then the scaling of $\tilde{u}$ can change and we can get a relevant scaling even in the region $1/z > 8\pi$. We numerically obtained that the initial condition $\frac{1}{8\pi z(k=1)} = 1.33$, $\tilde{u}(k=1) = 0.1$, $z_1(k=1) = 0$ gives such a trajectory.

IV. Summary

We derived evolution equations for the momentum dependent and field independent wavefunction renormalization in the case of the css regulator. The coupling $\tilde{z}_1$ scales in an irrelevant manner according to its mass dimension. The SG model is asymptotically safe and it is questionable whether the flow of $\tilde{z}_1$ gives infinity in the UV limit or not. This issue will be treated in a forthcoming publication.

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