Multifractal states at the Anderson metal-insulator transition

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Introduction
- The Anderson transition
- Intermediate statistics and multifractality
- Random matrix model: PBRM

Some new results using PBRM
- Spectral statistics, generalized dimensions
- Scattering, transport
- Magnetic impurities

Conclusions and outlook
Absence of Diffusion in Certain Random Lattices

P. W. Anderson
Bell Telephone Laboratories, Murray Hill, New Jersey
(Received October 10, 1957)

This paper presents a simple model for such processes as spin diffusion or conduction in the “impurity band.” These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.

Theoretical investigations of the electronic structure of magnetic and disordered systems.
Anderson’s model (1958)

- Hamiltonian
  \[ H = \sum_n \epsilon_n c_n^\dagger c_n + \sum_{n,m} V_{nm} c_n^\dagger c_m \]

- Energies \( \epsilon_n \) _uncorrelated_, random numbers from uniform (bimodal, Gaussian, Cauchy, etc.) distribution \( \Rightarrow W \)

- Nearest-neighbor „hopping” \( \Rightarrow V \) (symmetries: \( R, C, Q \))

![Diagram showing different regimes of \( V \) compared to \( W \): ballistic, diffusive, localized](image)
Anderson localization (BEC)
Anderson localization

**Very recent challenges**

- Bose-Einstein condensation (Ph. Bouyer)
- Ultrasound localization (van Tiggelen)
- Localization of EM field (H-J. Stöckmann)
- Quantum biology (G. Vattay, S. Kaufmann)
- Superradiance (F. Borgonovi)
- Enhanced superconducting $T_c$ (Feigelman)
- Quantum Chromo-Dynamics (T. G. Kovács)
- Etc...
Anderson localization

Billy et al. 2008

Hu et al. 2008

Sridhar 2000

Jendrzejewski et al. 2012
One-parameter scaling (1979)

Energy (time) scales: $E_{Th}$ and $\Delta$ ($\tau_D$ and $\tau_H$)

$$g = \frac{E_{th}}{\Delta} = \frac{\tau_H}{\tau_D}$$

Gell-Mann – Low function

$$\frac{1}{g} \frac{dg}{d \ln L} = \beta(g)$$

MIT: $d>2$ (O), $d>1$ (Sp)

<table>
<thead>
<tr>
<th></th>
<th>$W_c$</th>
<th>$\nu$</th>
</tr>
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<tbody>
<tr>
<td>O</td>
<td>16.45</td>
<td>1.57</td>
</tr>
<tr>
<td>U</td>
<td>18.32</td>
<td>1.43</td>
</tr>
<tr>
<td>Sp</td>
<td>18.93</td>
<td>1.375</td>
</tr>
</tbody>
</table>
Mobility edge (d=3)
Spectral statistics

- $W < W_c$: 
  - extended states
  - RMT-like

- $W > W_c$: 
  - localized states
  - Poisson-like
Dependence on symmetry parameter $\beta$

**superscaling** relation thru parameter $g$

\[ P_{g,\beta}(e^x) = \int_{-\infty}^{\infty} Q_{g,\beta}(e^{x-y}) W_{\beta}(e^y) \, dy \]

with $e^x \equiv s$ and $W_{\beta}(t)$ is the RMT limit

IV, Hofstetter, Pipek '99
Eigenstates at small and large $W$

Extended state
Weak disorder, midband

Localized state
Strong disorder, bandedge

(L=240) R.A. Römer
Multifractal eigenstate at the critical point

http://en.wikipedia.org/wiki/Metal-insulator_transition

(L=240) R.A.Römer
Multifractal eigenstate at the critical point

- Inverse participation ratio

$$I_\alpha(q) = \int_\Omega d^d r \, |\psi_\alpha(r)|^{2q} \propto L^{-d_q(q-1)}$$

- higher precision
- scaling with $L$

- Box-counting technique
  - fixed $L$
  - „state-to-state” fluctuations

- PDF analysis
Multifractal eigenstate at the critical point
Multifractal eigenstate at the critical point
Multifractal eigenstate at the critical point

Do these states exist at all?

Yes
Multifractal states in reality

Quantum Hall Transition in Real Space: From Localized to Extended States
K. Hashimoto,1,2,3,* C. Sohrmann,4 J. Wiebe,1 T. Inaoka,5 F. Meier,1,† Y. Hirayama,2,3 R. A. Römer,4 R. Wiesendanger,1 and M. Morgenstern5,7

LDOS at the QH transition observed on a Cs layer evaporated over an n-InAs(100) surface
Multifractal states in reality

Visualizing Critical Correlations Near the Metal-Insulator Transition in Ga$_{1-x}$Mn$_x$As
Anthony Richardella, et al.
Science 327, 665 (2010);
DOI: 10.1126/science.1183640

LDOS fluctuations in the vicinity of the metal-insulator transition Ga$_{1-x}$Mn$_x$As
Multifractality in general

- Turbulence (Mandelbrot)
- Time series – signal analysis
  - Earthquakes
  - ECG, EEG
  - Internet data traffic modelling
  - Share, asset dynamics
  - Music sequences
  - etc.

- Complexity
  - Human genome
  - Strange attractors
  - etc.

Common features
- self-similarity across many scales,
- broad PDF
- multiplicative processes
- rare events
Multifractality in general

- Very few analytically known $D(q)$
  - binary branching process
  - 1d off-diagonal Fibonacci sequence
  - Baker’s map
  - etc

- Numerical simulations
  - Perturbation series (Giraud 2013)
  - Renormalization group - NL0M – SUSY (Mirlin)

- Heuristic arguments
Numerical multifractal analysis

Parametrization of wave function intensities \( |\psi_j|^2 \sim L^{-\alpha} \)

The set of points where \( |\psi_j|^2 \) scales with \( \alpha \) \( N_\alpha \sim L^{f(\alpha)} \)

\[ \mu_k(\ell) = \sum_{j \in \text{box}_k} |\psi_j|^2 \Rightarrow I_q(\ell) = \sum_k [\mu_k(\ell)]^q \]

\[ I_q(\ell) \propto \left( \frac{\ell}{L} \right)^{\tau(q)} \Rightarrow \left\{ \begin{array}{l} \alpha_q = \tau'(q) \\ f(\alpha_q) = q\alpha_q - \tau(q) \end{array} \right. \]

Scaling:
- Box size: \( \ell \rightarrow 0 \)
- System size: \( L \rightarrow \infty \)

Averaging:
- Typical: \( \exp\{\ln I_q(\ell)\} \)
- Ensemble: \( \langle I_q(\ell) \rangle \)
Numerical multifractal analysis

Parametrization of wave function intensities \( |\Psi_j|^2 \sim L^{-\alpha} \)

The set of points where \( |\Psi_j|^2 \) scales with \( \alpha \) \( \mathcal{N}_\alpha \sim L^{f(\alpha)} \)

- \( |\Psi_j|^2 \leq 1 \implies \alpha \geq 0 \)
- \( f(\alpha) \) convex \( f(\alpha_0 > d) = d \)
- \( f(\alpha_1) = \alpha_1 \)
- Symmetry (Mirlin, et al. 06)
  \[ \alpha_q + \alpha_{1-q} = 2d \]
  \[ f(2d - \alpha) = f(\alpha) + d - \alpha \]
Numerical multifractal analysis

Generalized inverse participation number, Rényi-entropies

\[ \langle I_q \rangle \sim \begin{cases} \frac{L^0}{L} & \tau(q) \leq 1 \\ \frac{L^{-\tau(q)}}{L^{-d(q-1)}} & \end{cases} \]

\[ R_q = \frac{1}{1 - q} \ln \langle I_q \rangle \sim D_q \ln L \]

Mass exponent, generalized dimensions

\[ \tau_q = d(q - 1) + \Delta_q \equiv D_q(q - 1) \implies \Delta_q = \Delta_{1-q} \]

Wave function statistics

\[ \mathcal{P}(\ln|\psi|^2) \sim L^f \left( \frac{\ln|\psi|^2}{\ln L} \right)^{-d} \]

parabolic \( f(\alpha) \)
log-normal \( \mathcal{P}(x) \)
Numerical multifractal analysis

\[ \mu_k(\ell) = \sum_{j \in \text{box}_k} |\Psi_j|^2 \]

\[ \alpha = \frac{\ln \mu}{\ln \lambda} \quad \text{if} \quad \lambda = \frac{\ell}{L} \]

\[ P(\alpha) \sim \lambda^{d-f(\alpha)} \]

Rodriguez et al. 2010
Unusual features at the transition

- Interplay of eigenvector and spectral statistics
  - $q=2$, Chalker et al. ’95
  - $q=1$, Bogomolny 2011

- Anomalous diffusion
  - Huckestein et al. ’97

- LDOS and wave function fluctuations
  - Huckestein et al. ’97

- Correlation dimension
  - Kravtsov and Cuevas ‘07

\[ \langle |\Psi_n(r)|^2 |\Psi_m(r)|^2 \rangle \propto |E_n - E_m|^{-[1-(D_2/d)]} \]
Correlations at the transition

Cross-correlations of multifractal eigenstates

\[ C_{nm} = L^d \int d^d r \left| \langle \psi_n(r) |^2 \psi_m(r) |^2 \rangle \right| = \begin{cases} (E_c / [\max(\{\omega_{nm}, \Delta\})])^\gamma, & 0 < |\omega_{nm}| < E_c, \\ (E_c / |\omega_{nm}|)^2, & E_c < |\omega_{nm}| \end{cases} \]

Auto-correlation of multifractal eigenstates

\[ C_{r_1, r_2} = L^{2d} \left| \langle |\psi_n(r_1) |^2 |\psi_n(r_2) |^2 \rangle_n \right| = \left( \frac{L}{|r_1 - r_2|} \right)^d, \quad 0 < |r_1 - r_2| < L \]

where

\[ \gamma = 2(\alpha_0 - d)/d = (1 - d_2/d) \]
Scattering: system + lead

- Scattering matrix

\[ S(E) = e^{i\Phi(E)} = 1 - 2i\pi W^T (E1 - \mathcal{H}_{\text{eff}})^{-1} W \]

\[ \mathcal{H}_{\text{eff}} = H - i\pi WW^T \]

\[ W_n = w_0 \delta_{nn_0} \]

- Wigner-Smith delay time

\[ \tau(E = 0) = \left. \frac{d\Phi(E)}{dE} \right|_{E=0} = -2\text{ImTr}(E - \mathcal{H}_{\text{eff}})^{-1} \bigg|_{E=0} \]

- Resonance widths: eigenvalues of \( \mathcal{H}_{\text{eff}} \), poles of \( S(E) \)
Scattering, open system

\[ P(\tilde{\Gamma}) \sim \tilde{\Gamma}^{-(1 + 1/3)}, \quad \tilde{\Gamma} = \Gamma/\Delta \]

\[ P(\tilde{\tau}) \sim \tilde{\tau}^{-2.5}, \quad \tilde{\tau} = \tau M \Delta \]

Find the transition with a stopwatch!

Kottos and Weiss '02; Weiss, et al. '06
Effect of multifractality

- Scattering
- Conductance, shot-noise
- LDOS fluctuations, dynamics
- Entanglement
- Magnetic impurities
Effect of multifractality

Generalize!

Take the model of the model!

PBRM
**PBRM**: Power-law Band Random Matrix

- **model**: $N \times N$ matrix, $\langle H_{ij} \rangle = 0$

\[
\langle (H_{ij})^2 \rangle = \frac{1}{\beta} \left( 1 + \left[ \frac{\sin\left( \frac{\pi}{N} |i-j| \right)}{\frac{\pi}{N} b} \right]^{2\mu} \right)^{-1}
\]

- **asymptotically**: $(1 \ll |i-j| \ll N - 1)$

\[
\langle (H_{ij})^2 \rangle \sim \left( \frac{b}{|i-j|} \right)^{2\mu}
\]

- **free parameters**: $\mu$ and $b$

Mirlin, *et al.* '96, Mirlin '00
PBRM

- for $\mu < 1 \Rightarrow \text{RMT, as if } \mu \to 0$
  - for $1/2 < \alpha < 1 \Rightarrow \text{similar to metal with } d = 1/(\mu - 1/2)$

- for $\mu > 1 \Rightarrow \text{BRM } \Rightarrow \text{Poisson, as if } \mu \to \infty$
  - for $\mu > 3/2 \Rightarrow \text{power law localization with exponent } \mu \text{ (cf. Yeung-Oono '87)}$

- for $\mu = 1 \Rightarrow \text{criticality (cf. Levitov '90)}$
  - continuous line of transitions: $b$

Mirlin, et al. '96, Mirlin '00
PBPM at criticality ($\mu = 1$)

- $b \gg 1$ weak multifractality
  $D_2 \rightarrow 1 \quad \chi \rightarrow 0$

- $b \ll 1$ strong multifractality
  $D_2 \rightarrow 0 \quad \chi \rightarrow 1$

\[ \chi \sim (1 + a \chi b)^{-1} \]

Mirlin, et al. '96, Mirlin '00
PBPRM at criticality \((b = 1)\)

semi-Poisson statistics qualitatively valid

IV and Braun (2000)
joint distribution

state-to-state fluctuation

\[ \beta = 1 \]

\[ \beta = 2 \]

\[ \langle D_2 \rangle = \begin{cases} 
1 - (\pi b \beta)^{-1}, & b \gg 1 \\
2b \beta, & b \ll 1 
\end{cases} \]

\[ P(D_2) \]
Generalized dimensions vs. $b$

$$D_q \approx \left[ 1 + \left( a_q b \right)^{-1} \right]^{-1}$$

$$D_q' \approx \frac{q D_q}{q' + (q - q') D_q}$$
PBGM at criticality \((0.5 < q < 5)\)

General relations

\[
\frac{qD_q}{1 - D_q} \approx \text{const} \sim b
\]

Spectral statistics and \(D_q\)

\[
\chi \approx \frac{1 - D_q}{1 + (q - 1)D_q}
\]

\[
e.g.: \quad D_2 \approx \frac{D_1}{2 - D_1} \approx \frac{1 - \chi}{1 + \chi}
\]
Higher dimensions, $q < 0.5$

Replace $D_q \rightarrow \frac{D_q}{d}$

For $q < 0.5$ using $\Delta_q = \Delta_{1-q}$

$$D_q \approx \frac{1 - 2q}{1 - q} + \frac{q}{1 - q} \left( \frac{D_1}{1 + q(D_1 - 1)} \right)$$

$$\chi \approx \frac{1 - D_q}{q(2 - D_q)}$$

JAMB és IV (2013)
Scattering: PBRM + 1 lead

- JA Méndez-Bermúdez – Kottos ‘05
  Ossipov – Fyodorov ‘05:

\[
\langle \tau^{-q} \rangle \propto L^{-qD_{q+1}}
\]

\[
\tilde{P}_w(\tau_w) = \frac{1}{\tau_w^3} P_y\left(\frac{1}{\tau_w}\right)
\]

Measure the multifractality using a stopwatch!

- JA Méndez-Bermúdez – IV 06:

\[
\tau^{\text{typ}} = \exp \langle \ln \tau \rangle
\]

\[
\tau^{\text{typ}} \propto L^{D_1}
\]

\[
\Gamma^{\text{typ}} = \exp \langle \ln \Gamma \rangle
\]

\[
\Gamma^{\text{typ}} \propto L^{-(2-D_2)}
\]
Scattering exponents

Wigner-Smith delay time

\[ \exp(\ln \tau) \sim L^{\sigma_\tau} \]

\[ \langle \tau^{-q} \rangle \sim L^{-\sigma_q} \]

\[ \sigma_\tau = 1 - \chi \]

\[ \sigma_q = \frac{q(1 - \chi)}{1 + q\chi} \]
Critical Metal Phase at the Anderson Metal-Insulator Transition with Kondo Impurities

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3Elméleti Fizika Tanszék, Budapesti Műszaki és Gazdaságtudományi Egyetem, H-1521 Budapest, Hungary
(Received 27 October 2008; published 14 September 2009)
Kondo effect in metals (1964)

below $T < T_K$ spin-flip scattering
singlet ground-state
Kondo-screening
Kondo effect in disordered metals

**Metal**
- Weak disorder → extended states
- Full screening, **no** free magnetic moments

**Insulator**
- Strong disorder → localized states,
- No screening → free magnetic moments exist

**Criticality** → multifractality
- Screening? Frozen Kondo-effect?
- Are there any free magnetic moments?
Kondo effect in disordered metals

Local \( T_K \) broad, bimodal \( P(T_K) \)

1-loop (Nagaoka – Suhl):

Unscreened (free) magnetic moments exist:

\[
T_K = 0 \quad \text{ha} \quad J < J_c
\]

Weekly disordered conductor:

\[
J_c \sim D/\ln N \to 0
\]

\( D \) is the bandwidth

Insulator:

\[
J_c \sim D/\ln N_I, \quad N_I = D/\Delta_I
\]

\[
\Delta_I = (\nu \xi^d)^{-1}
\]
Kondo effect at the critical point

log-normally distributed wave functions

\[
P(\alpha_l) \sim L^{-d+f(\alpha_l)}
\]

\[
\alpha_l = -\frac{\ln |\psi_l(0)|^2}{\ln L}
\]

\[
f(\alpha) = d - \frac{(\alpha - \alpha_0)^2}{4(\alpha_0 - d)}
\]

joint distribution of wave function intensities

\[
P_{\alpha}(\alpha_l, \alpha_k) = L^{a_{lk}[f(\alpha_l)+f(\alpha_k)-2d]+b_{lk}\frac{(\alpha_l-\alpha_0)(\alpha_k-\alpha_0)}{2(\alpha_0-d)}}
\]

\[
a_{kl} = \left(1 + \sqrt{1 + 4b_{kl}^2}\right)/2 \quad b_{kl} = g_{kl}/(g_{kl}^2 - 1)
\]

energy correlations of eigenstates

\[
g_{lk} = \frac{\ln(|\omega_{lk}|/E_c)}{\ln L} \times \left\{ \begin{array}{ll} d^{-1}, & |\omega_{lk}| < E_c, \\ (\alpha_0 - d)^{-1}, & |\omega_{lk}| > E_c. \end{array} \right.
\]

\[
\omega_{lk} = E_l - E_k
\]

\[
J_c(\alpha) = \frac{\alpha - \alpha_0}{d} \left\{ 1 - \exp \left[ -\frac{(\alpha - \alpha_0)\sqrt{\ln L}}{2\sqrt{\alpha_0 - d}} \right] \right\}^{-1}
\]

\[
\frac{J_c(\alpha)}{D} = \frac{\alpha - \alpha_0}{d} \left\{ 1 - \exp \left[ -\frac{(\alpha - \alpha_0)\sqrt{\ln L}}{2\sqrt{\alpha_0 - d}} \right] \right\}^{-1}
\]
Kondo effect around the critical point

Free magnetic moments away from the critical point

\[ P_{FM} = n_{FM}/n_M = \text{Erfc} \left( \sqrt{\frac{\ln \xi}{2\gamma D}} \right) \]

\[ \gamma = 2(\alpha_0 - d)/d \]

\[ P_{FM} \sim (W - W_c)^{\kappa(J)} \]

\[ \kappa(J) = (\nu d/2\gamma)(J/D)^2 \]

\[ J_c^{(1)} = \sqrt{2\gamma D} \rightarrow n_{FM} = n_M/N \]
Symmetry class of the transition at $T=0K$

Symmetry-dependent critical point

In the case if

$$J < J^* = \left(2\sqrt{\gamma/d} - \gamma\right)D$$

and intermediate phase:

$$X_s = 1$$

Since

$$\frac{1}{\tau_{FM}} = \frac{1}{\tau_s^0} \left(\frac{\xi}{a_c}\right)^{-\frac{d}{2\gamma}}\left(\frac{1}{D} + \gamma\right)^2$$

hence

$$W_c(J) = W_c^0 \left[1 + \left(\frac{a_c}{Dc\tau_s^0}\right)^\eta(J)\right]$$

where

$$\eta(J) = \left[2\nu - \frac{d}{2\gamma} \left(\frac{J}{D} + \gamma\right)^2\right]^{-1}$$

$$X_s = \frac{\xi^2}{D\tau_s}$$

$$W_c^{\text{orth}} < W_c^{\text{unit}}$$
Summary and outlook

- Az Anderson-model és MIT
  - New experimental evidence
  - New topics involved

- Multifractal states in general
  - Random matrix models (PBRM)

- Outlook
  - Interacting particles (Coulomb) (cf. Mirlin, Kravtsov)
  - Decoherence – robust fidelity
  - Proximity (SC) – subgap spectrum
  - Topological insulators – ?
  - QCD ???????