Multifractal states at the Anderson metal-insulator transition

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Etienne Hofstetter (Mainz, London)

thanks to : OTKA, NKTH, AvH, DFG, CiC, Conacyt, etc.

Outline

- Introduction
  - The Anderson transition
  - Intermediate statistics and multifractality
  - Random matrix model: PBRM
- Some new results using PBRM
  - Spectral statistics, generalized dimensions
  - Scattering, transport
  - Magnetic impurities
- Conclusions and outlook

50 Years of Anderson Localization

Special Celebration

January 10th @ 7:30 pm

PHYSICAL REVIEW VOLUME 109, NUMBER 5 MARCH 1, 1958

Absence of Diffusion in Certain Random Lattices

P. W. ANDERSON
Bell Telephone Laboratories, Murray Hill, New Jersey
(Received October 10, 1957)

This paper presents a simple model for such processes as spin diffusion or conduction in the "impurity band." These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.

electronic structure of magnetic and disordered systems.

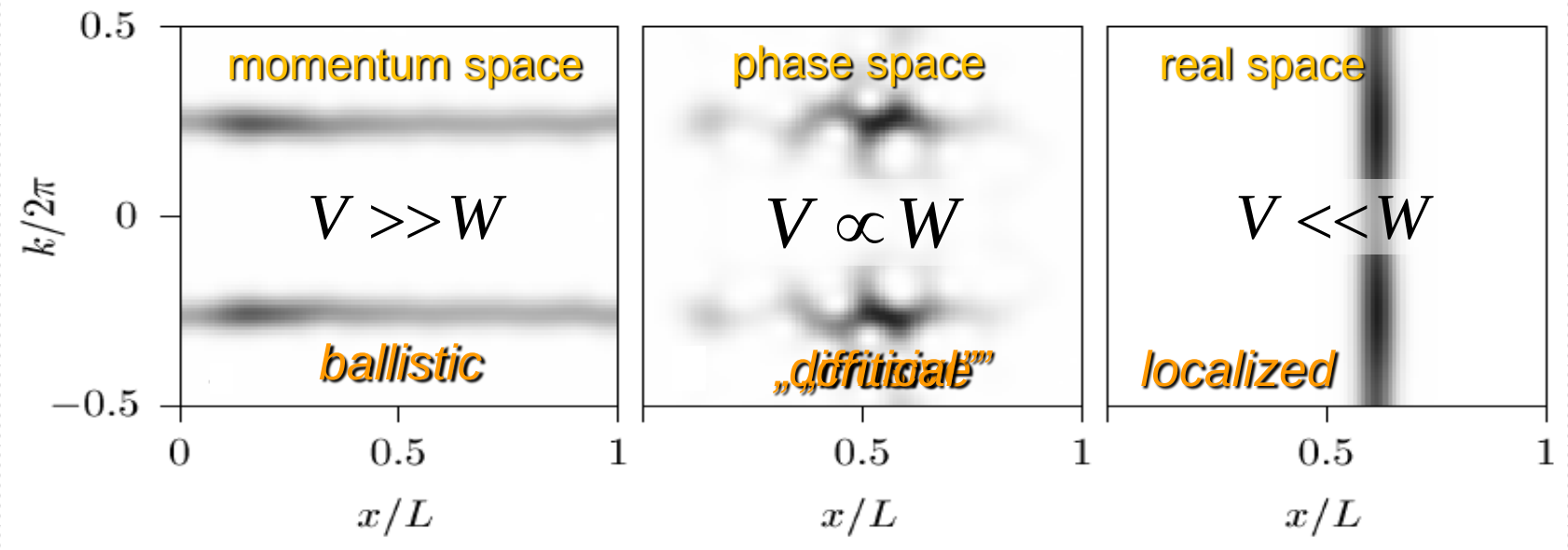
Anderson's model (1958)

- Hamiltonian

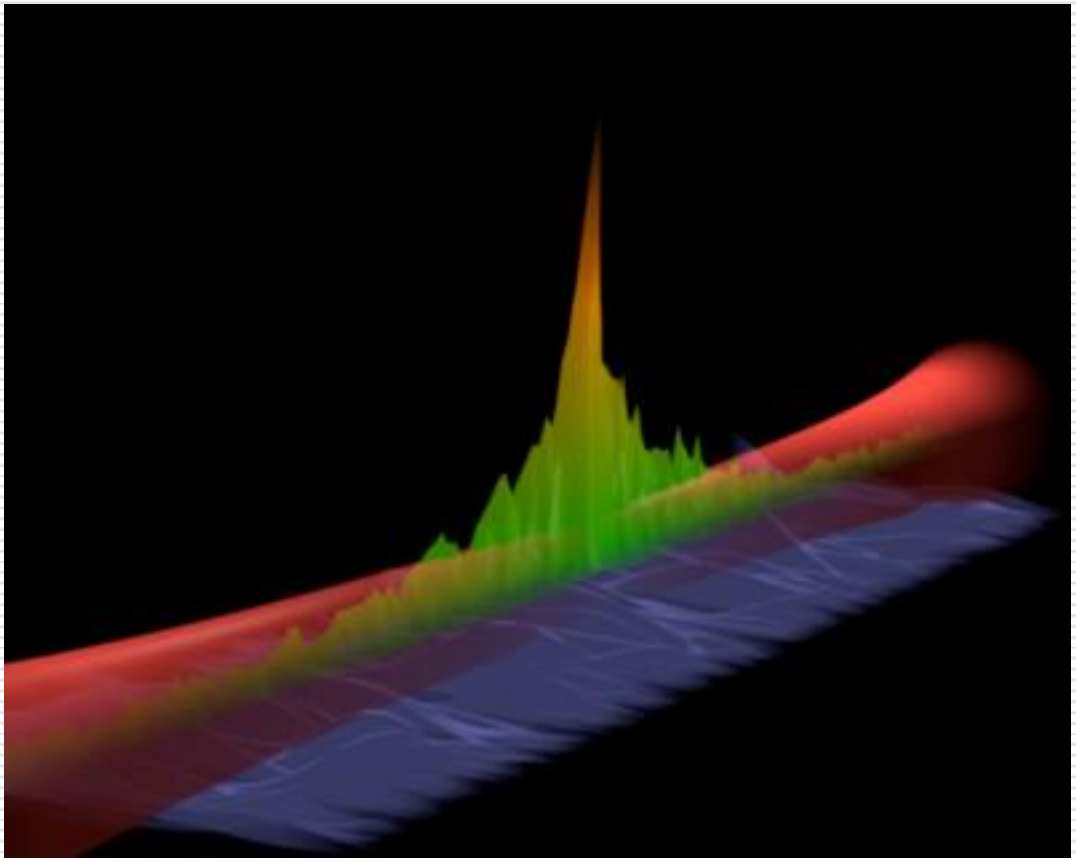
$$\mathcal{H} = \sum_n \epsilon_n c_n^\dagger c_n + \sum_{n,m} V_{nm} c_n^\dagger c_m$$

- Energies ϵ_n **<u>uncorrelated</u>**, random numbers from uniform (bimodal, Gaussian, Cauchy, etc.) distribution ⇒ W

- **<u>Nearest-neighbor</u>** „hopping" ⇒ V (symmetries: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$)



Anderson localization (BEC)



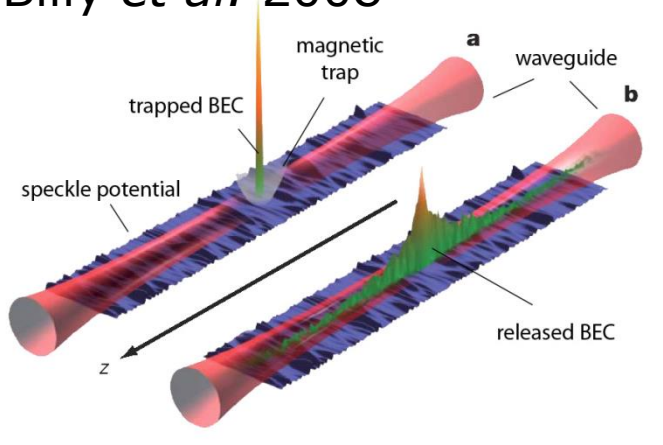
Anderson localization

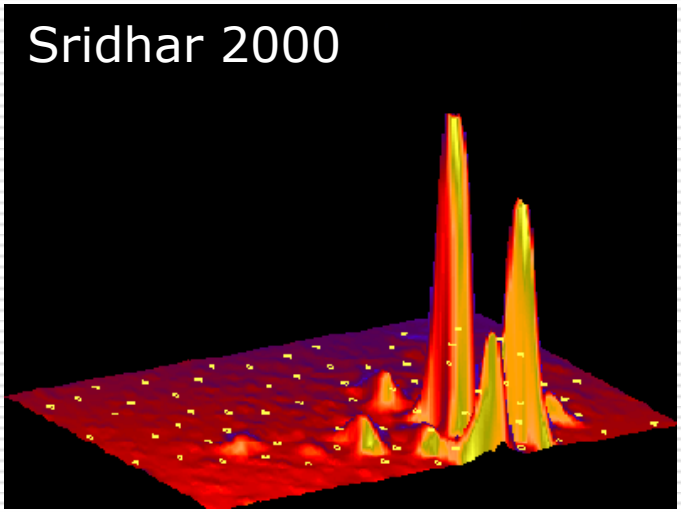
Very recent challenges

- Bose-Einstein condensation (Ph. Bouyer)
- Ultrasound localization (van Tiggelen)
- Localization of EM field (H-J. Stöckmann)
- Quantum biology (G. Vattay, S. Kaufmann)
- Superradiance (F. Borgonovi)
- Enhanced superconducting T_C (Feigelman)
- Quantum Chromo-Dynamics (T. G. Kovács)
- Etc…

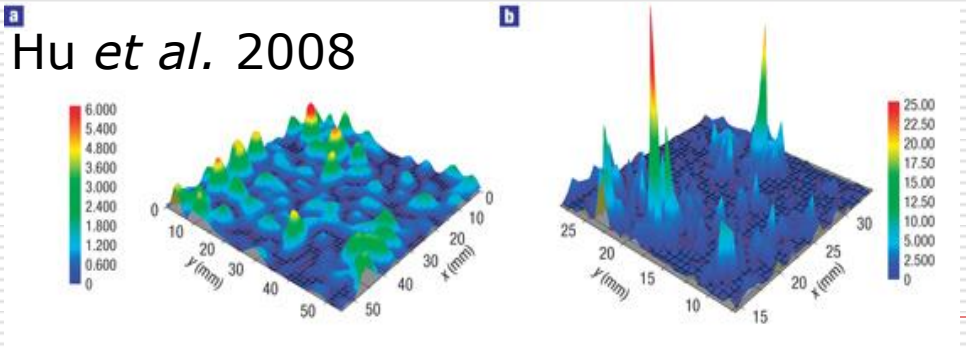
Anderson localization

Billy *et al.* 2008

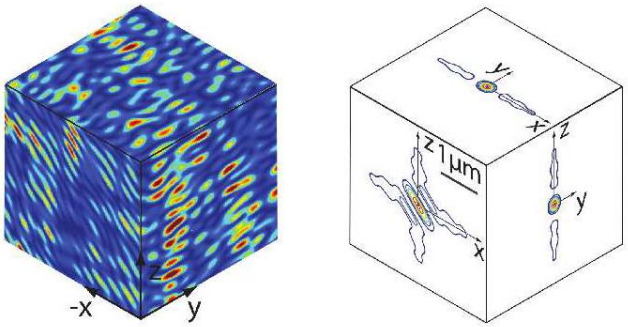




Hu *et al.* 2008

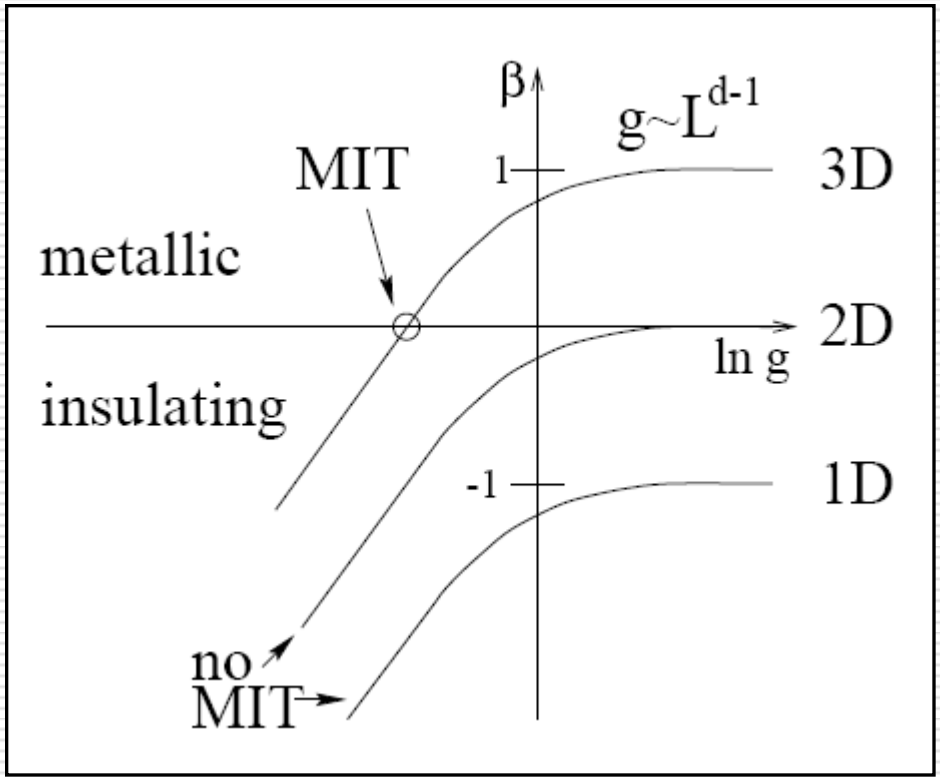


Jendrzejewski *et al.* 2012



One-parameter scaling (1979)

Energy (time) scales: E_{Th} and Δ (τ_D and τ_H) → $g = E_{th}/\Delta = \tau_H/\tau_D$



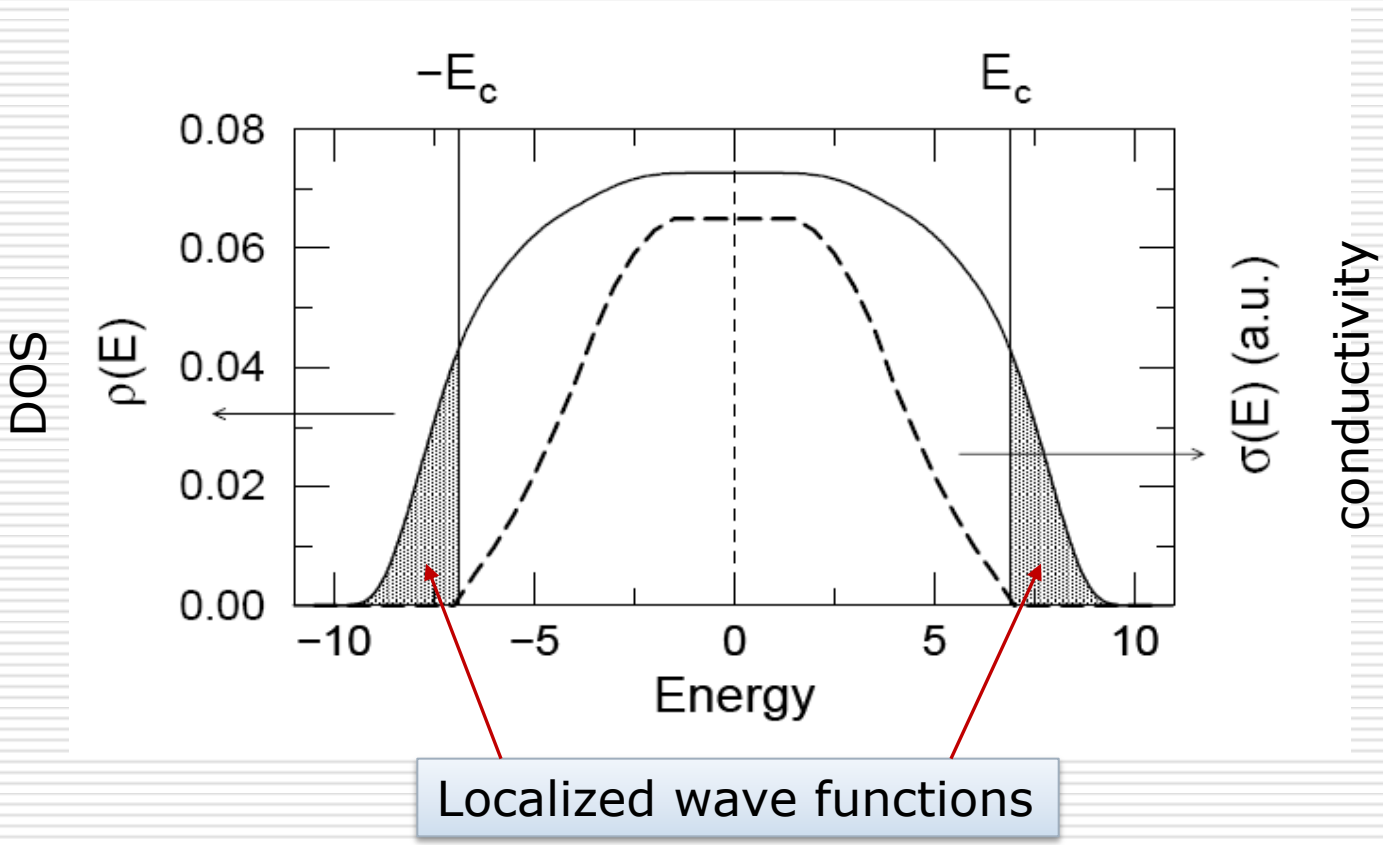
Gell-Mann – Low function

$$\frac{1}{g}\frac{dg}{d\ln L} = \beta(g)$$

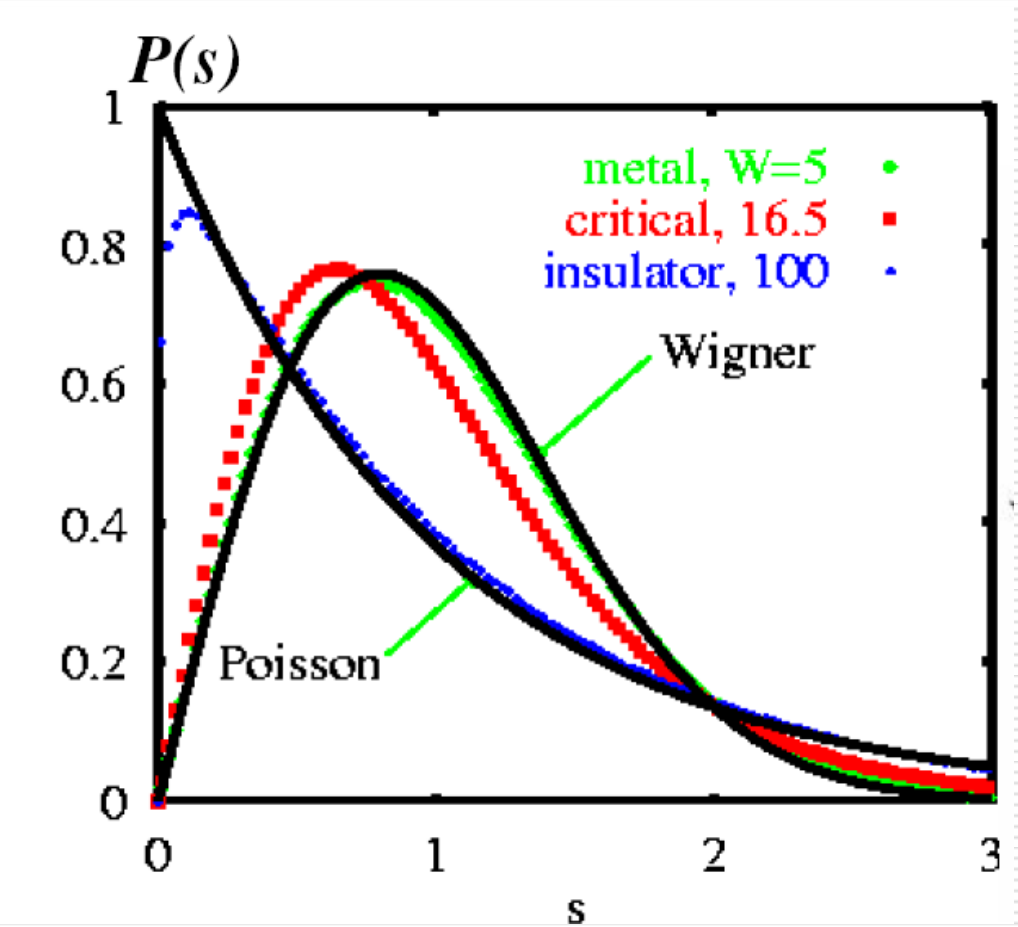
MIT: $d>2$ (O), $d>1$ (Sp)

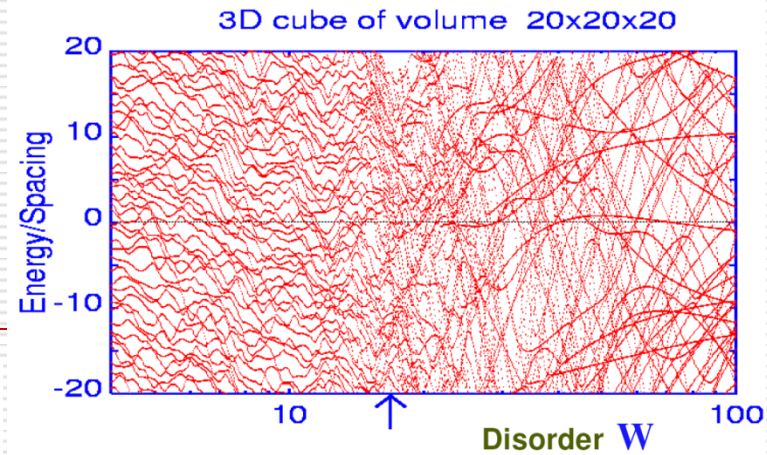
	W_C	ν
O	16.45	1.57
U	18.32	1.43
Sp	18.93	1.375

Mobility edge (d=3)



Spectral statistics





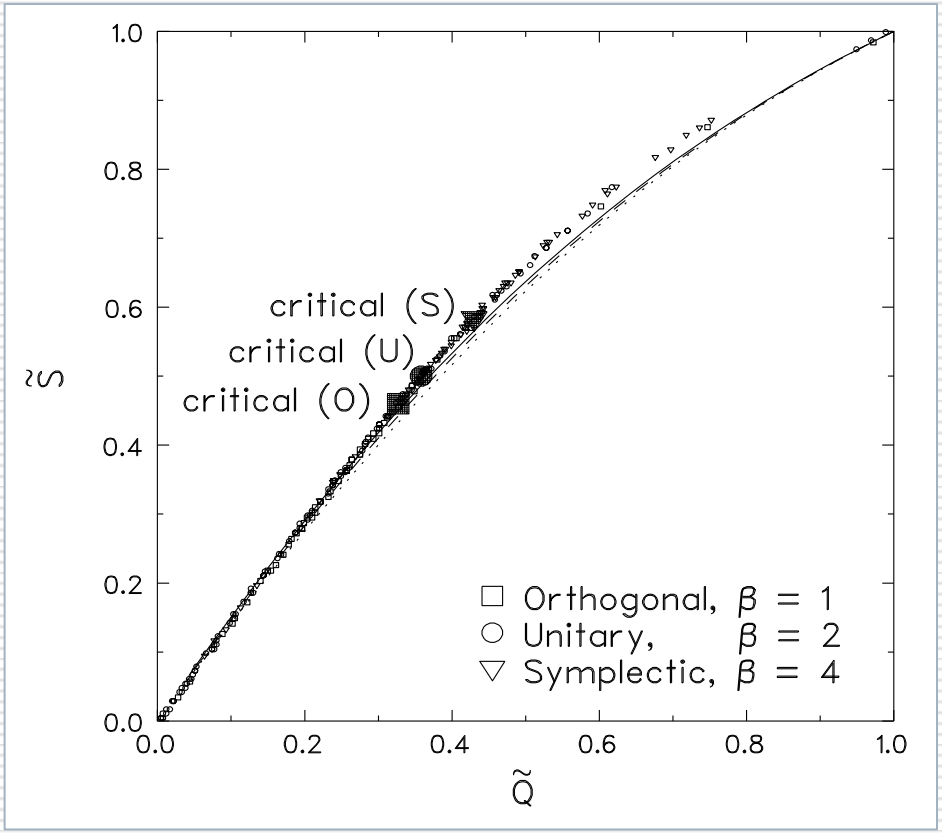
- $W < W_c$
  - extended states
  - RMT-like
- $W > W_c$
  - localized states
  - Poisson-like
- 

Anderson - MIT

- Dependence on symmetry parameter β
  superscaling relation
  thru parameter g

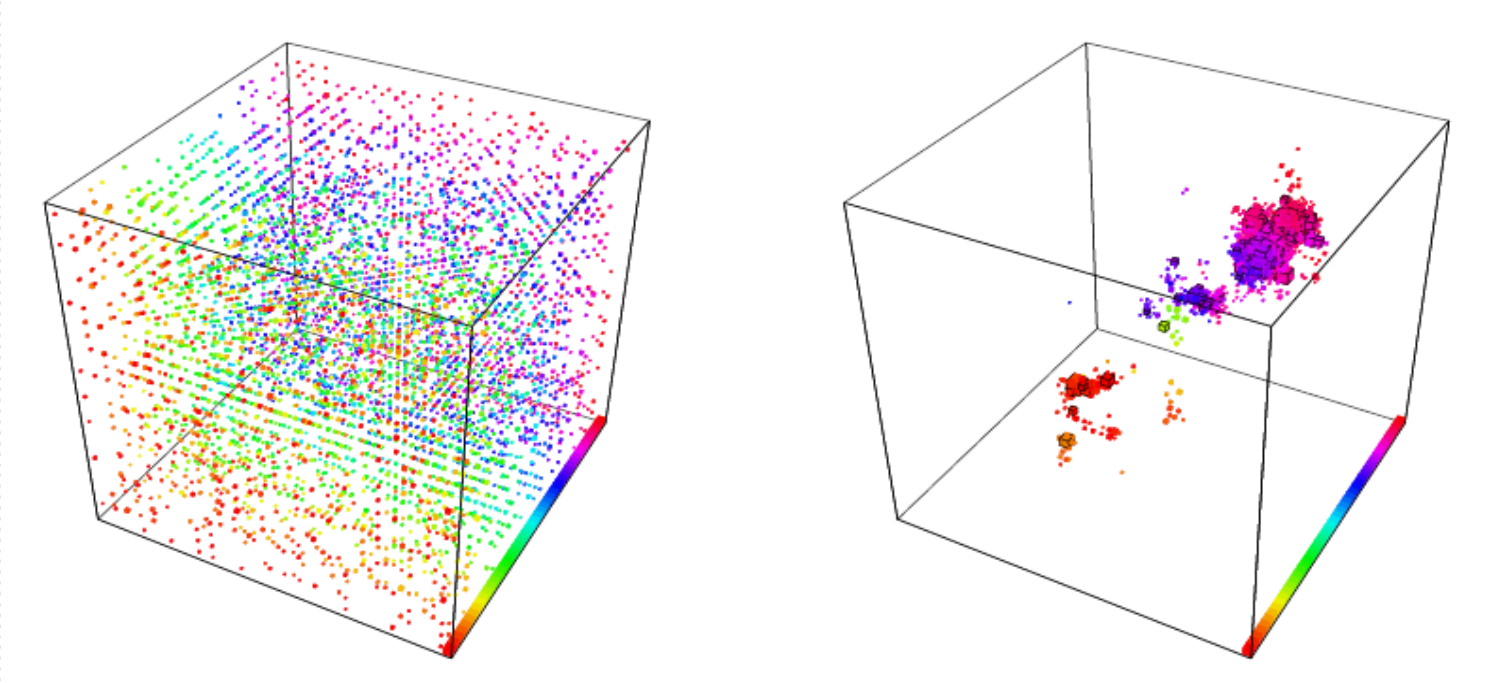
$$\mathcal{P}_{g,\beta}(e^x) = \int_{-\infty}^{\infty} \mathcal{Q}_{g,\beta}(e^{x-y})\,\mathcal{W}_\beta(e^y)\,dy$$

with $e^x \equiv s$ and
$\mathcal{W}_\beta(t)$ is the RMT limit



IV, Hofstetter, Pipek '99

Eigenstates at small and large *W*

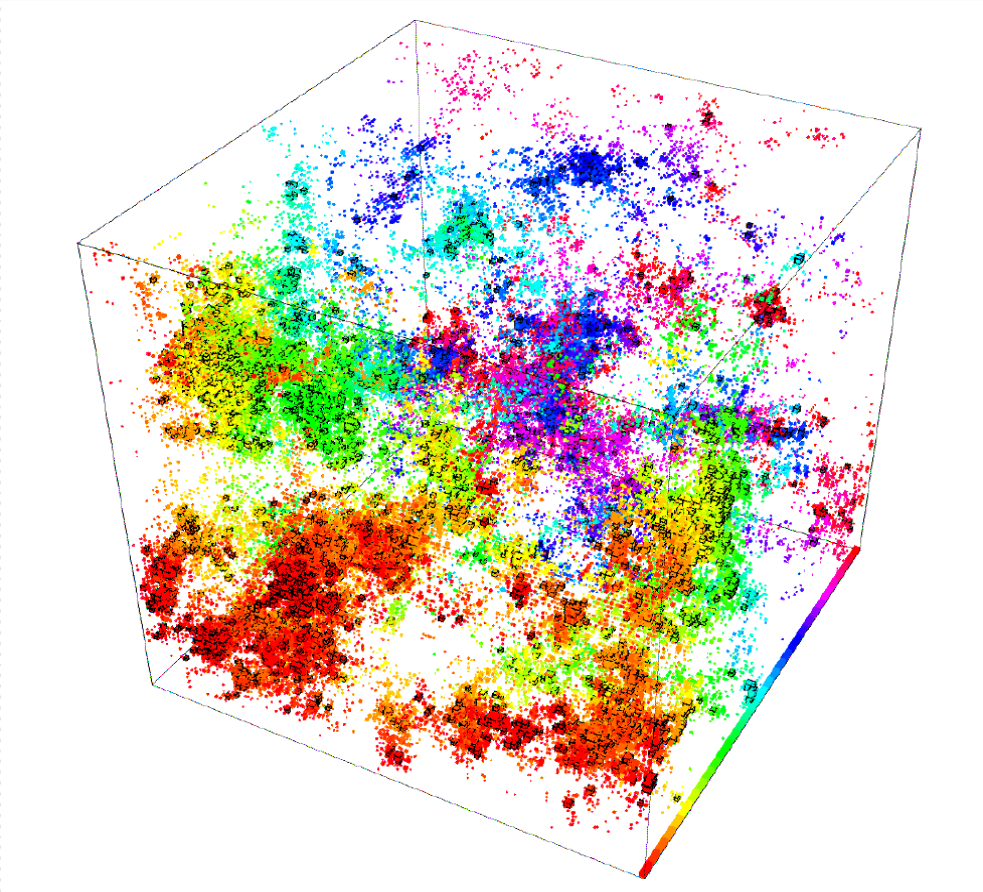


Extended state
Weak disorder, midband

Localized state
Strong disorder, bandedge

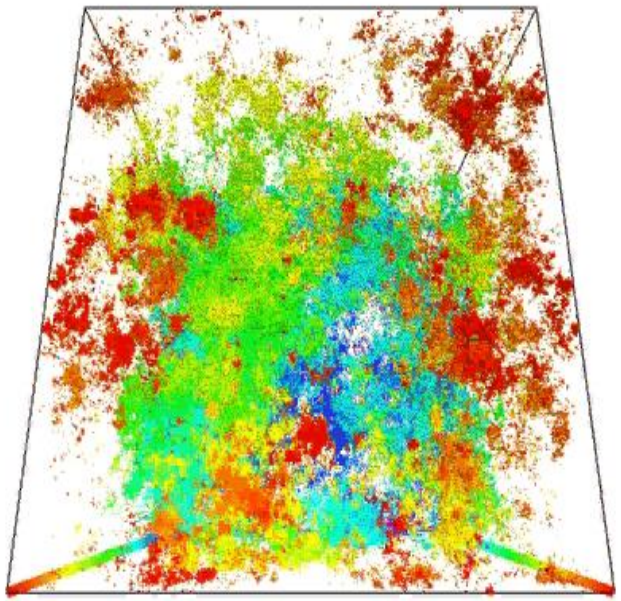
(L=240) R.A.Römer

Multifractal eigenstate at the critical point



http://en.wikipedia.org/wiki/Metal-insulator_transition (L=240) R.A.Römer

Multifractal eigenstate at the critical point



- Inverse participation ratio

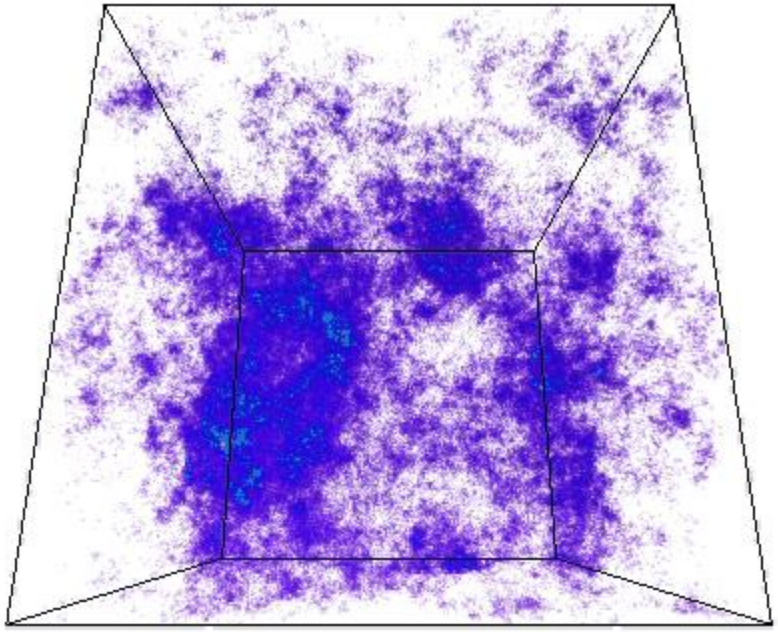
$$I_\alpha(q) = \int_\Omega d^d r \left|\psi_\alpha(\mathbf{r})\right|^{2q} \propto L^{-d_q(q-1)}$$

  - higher precision
  - scaling with L

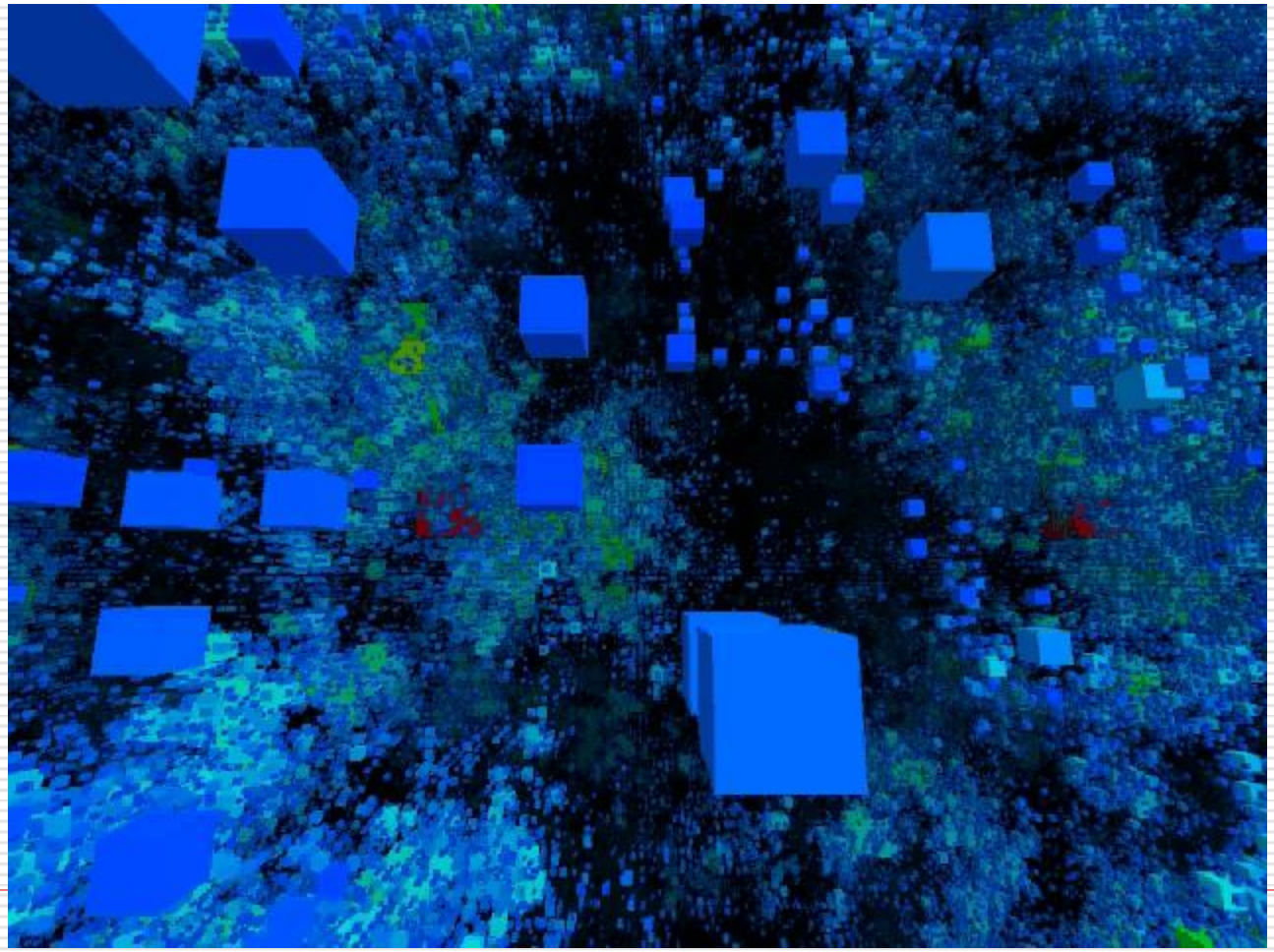
- Box-counting technique
  - fixed L
  - „state-to-state" fluctuations

- PDF analyzis

Multifractal eigenstate at the critical point



Multifractal eigenstate at the critical point



Multifractal eigenstate at the critical point

Do these states exist at all?

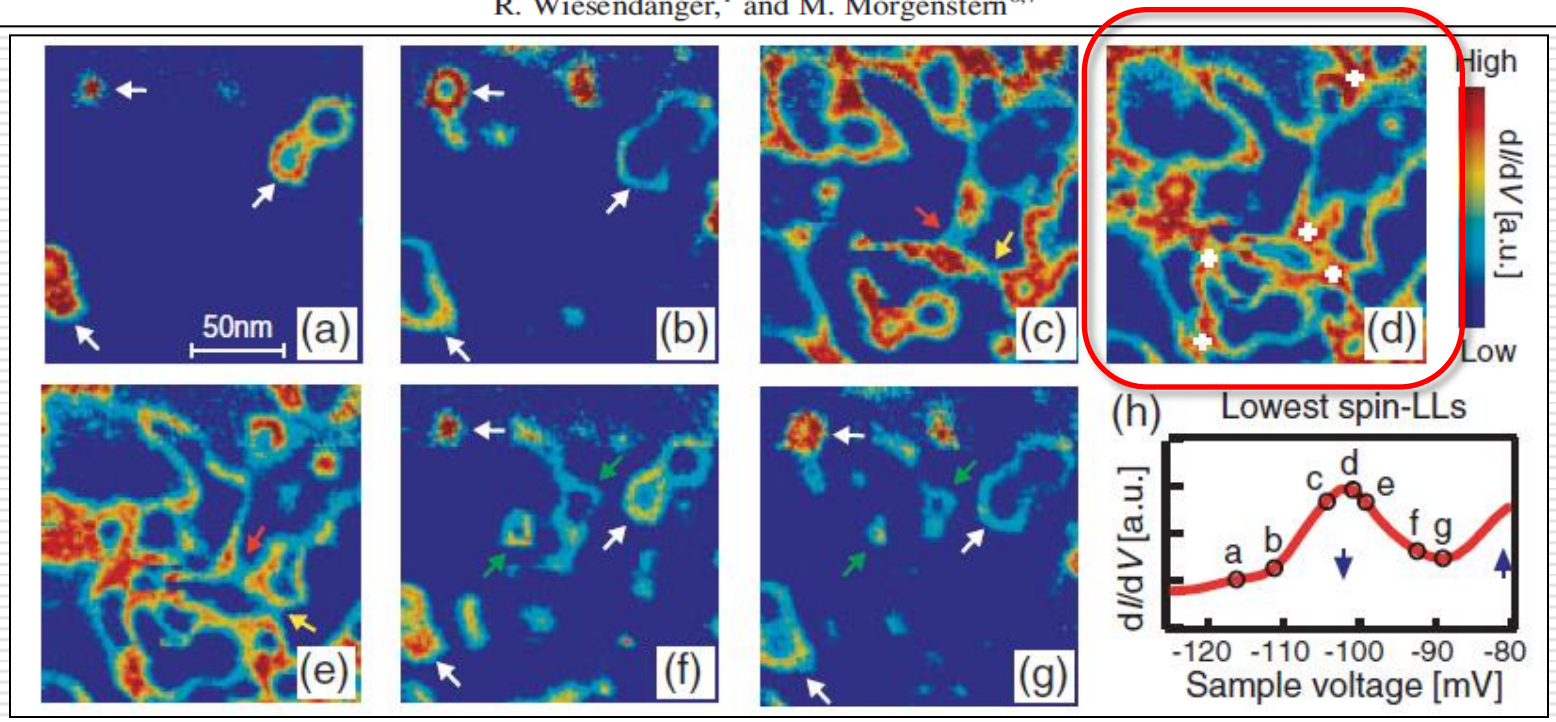
Yes

Multifractal states in reality

PRL 101, 256802 (2008) PHYSICAL REVIEW LETTERS week ending 19 DECEMBER 2008

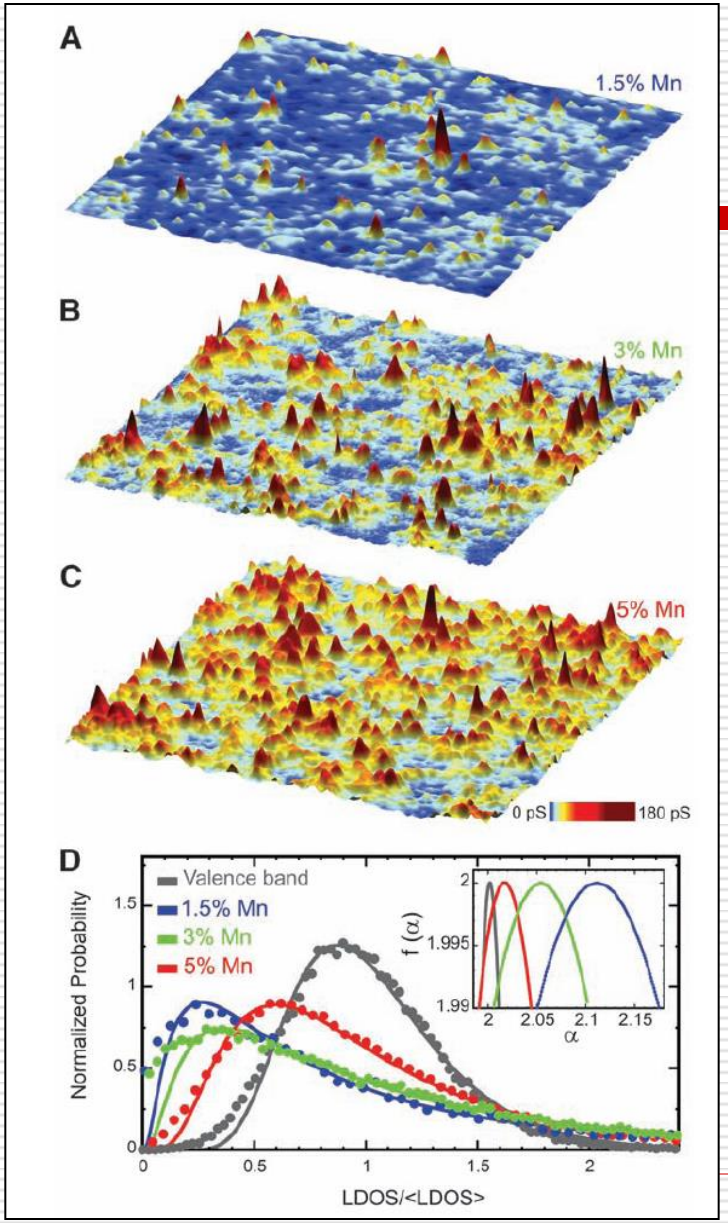
Quantum Hall Transition in Real Space: From Localized to Extended States

K. Hashimoto,[1,2,3,*] C. Sohrmann,[4] J. Wiebe,[1] T. Inaoka,[5] F. Meier,[1,†] Y. Hirayama,[2,3] R. A. Römer,[4] R. Wiesendanger,[1] and M. Morgenstern[6,7]



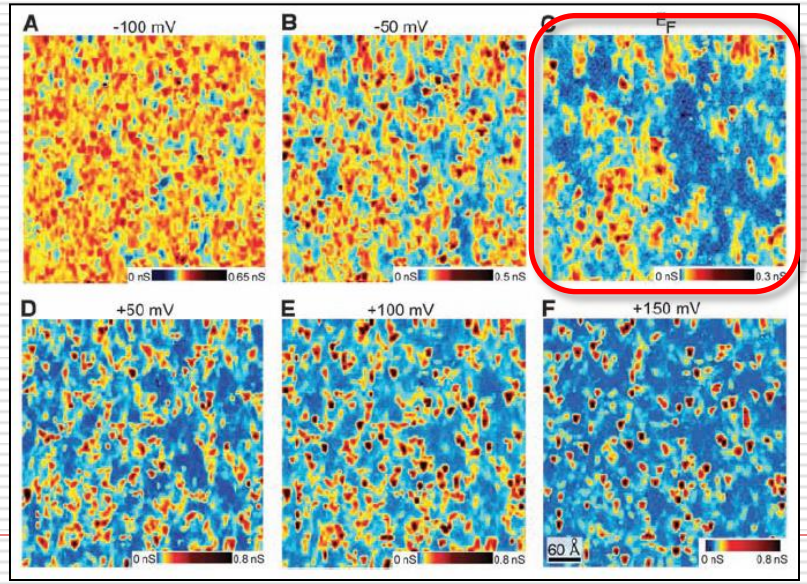
LDOS at the QH transition observed on a Cs layer evaporated over an n-InAs(100) surface

Multifractal states in reality



Visualizing Critical Correlations Near the Metal-Insulator Transition in Ga 1-xMnxAs
Anthony Richardella, *et al.*
Science **327**, 665 (2010);
DOI: 10.1126/science.1183640

LDOS fluctuations in the vicinity of the metal-insulator transition $\mathbf{Ga_{1-x}Mn_xAs}$



Multifractality in general

- Turbulence (Mandelbrot)
- Time series – signal analysis
  - Earthquakes
  - ECG, EEG
  - Internet data traffic modelling
  - Share, asset dynamics
  - Music sequences
  - etc.
- Complexity
  - Human genome
  - Strange attractors
  - etc.

Common features

- self-similarity across many scales,
- broad PDF
- muliplicative processes
- rare events

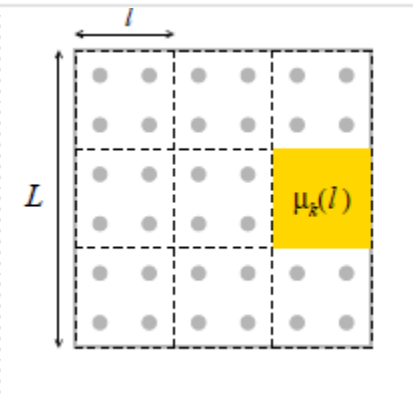
Multifractality in general

- Very few analytically known $D(q)$
  - binary branching process
  - 1d off-diagonal Fibonacci sequence
  - Baker's map
  - etc
- Numerical simulations
  - Perturbation series (Giraud 2013)
  - Renormalization group - NLσM – SUSY (Mirlin)
- Heuristic arguments

Numerical multifractal analysis

Parametrization of wave function intensities $\left|\Psi_j\right|^2 \sim L^{-\alpha}$

The set of points where $\left|\Psi_j\right|^2$ scales with α $\mathcal{N}_\alpha \sim L^{f(\alpha)}$



$$\mu_k(\ell) = \sum_{j \in \mathrm{box}_k} \left|\Psi_j\right|^2 \Rightarrow I_q(\ell) = \sum_k [\mu_k(\ell)]^q$$

$$I_q(\ell) \propto \left(\frac{\ell}{L}\right)^{\tau(q)} \Rightarrow \begin{cases} \alpha_q = \tau'(q) \\ f(\alpha_q) = q\alpha_q - \tau(q) \end{cases}$$

Scaling:

- Box size: $\ell \longrightarrow 0$
- System size: $L \longrightarrow \infty$

Averaging:

- Typical: $\exp\langle \ln I_q(\ell) \rangle$
- Ensemble: $\langle I_q(\ell) \rangle$

Numerical multifractal analysis

Parametrization of wave function intensities $|\Psi_j|^2 \sim L^{-\alpha}$

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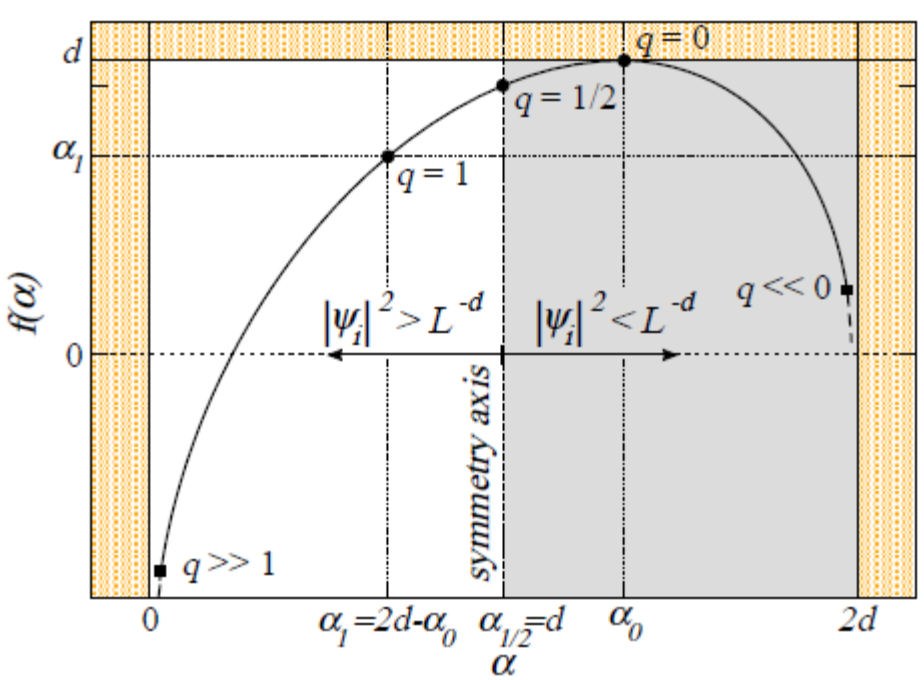


- $|\Psi_j|^2 \leq 1 \Longrightarrow \alpha \geq 0$
- $f(\alpha)$ convex $f(\alpha_0 > d) = d$
- $f(\alpha_1) = \alpha_1$
- Symmetry (Mirlin, *et al.* 06)
  $\alpha_q + \alpha_{1-q} = 2d$
  $f(2d - \alpha) = f(\alpha) + d - \alpha$

Numerical multifractal analysis

Generalized inverse participation number, Rényi-entropies

$$\langle I_q \rangle \sim \begin{cases} L^0 \\ L^{-\tau(q)} \\ L^{-d(q-1)} \end{cases} \qquad R_q = \frac{1}{1-q} \ln \langle I_q \rangle \sim D_q \ln L$$

Mass exponent, generalized dimensions

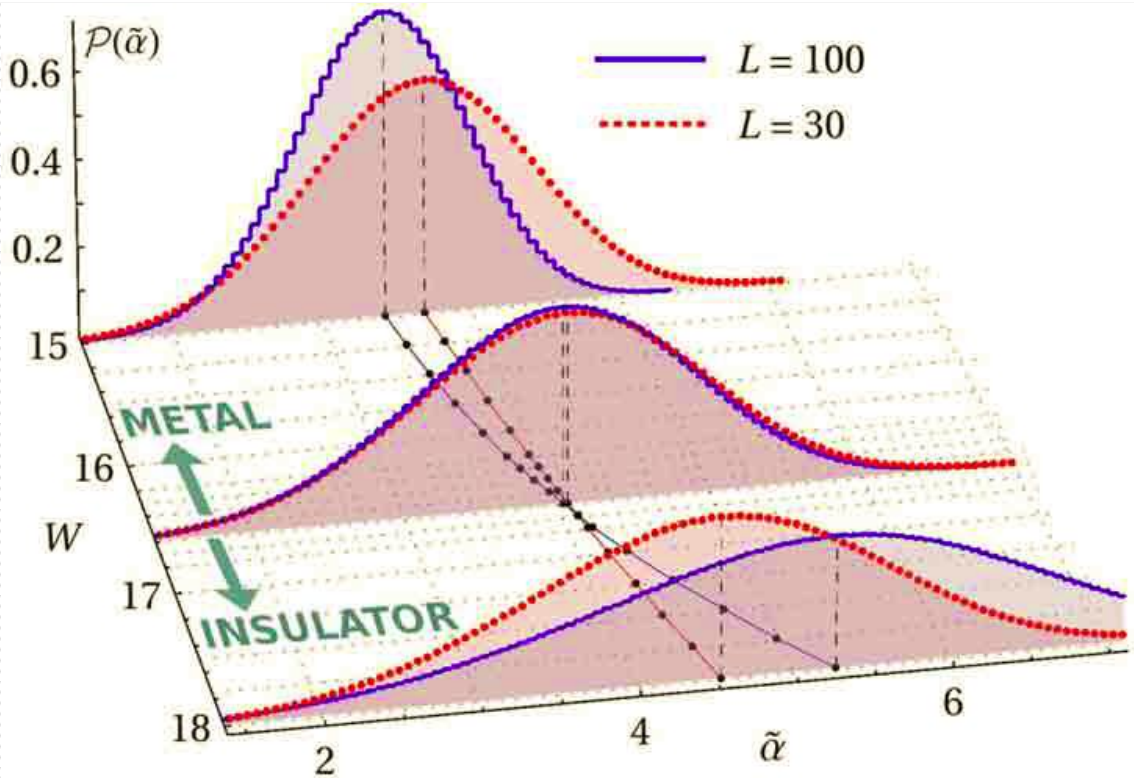
$$\tau_q = d(q-1) + \Delta_q \equiv D_q(q-1) \implies \Delta_q = \Delta_{1-q}$$

Wave function statistics

$$\mathcal{P}(\ln|\psi|^2) \sim L^{f\left(\frac{\ln|\psi|^2}{\ln L}\right)-d}$$

parabolic $f(\alpha)$
log-normal $\mathcal{P}(x)$

Numerical multifractal analysis



$$\mu_k(\ell) = \sum_{j \in \mathrm{box}_k} |\Psi_j|^2$$

$$\alpha = \frac{\ln \mu}{\ln \lambda} \quad \text{if} \quad \lambda = \frac{\ell}{L}$$

$$\mathcal{P}(\alpha) \sim \lambda^{d-f(\alpha)}$$

Rodriguez *et al.* 2010

Unusual features at the transition

- Interplay of eigenvector and spectral statistics
  - *q=2*, Chalker *et al.* '95
  - *q=1*, Bogomolny 2011

D_2

$$q\chi + \frac{D_q}{d} = 1$$

- Anomalous diffusion
  - Huckestein *et al*. '97

$$P(t) \propto t^{-D_2/d}$$

- LDOS and wave function fluctuations
  - Huckestein *et al.* '97

$$D_q^{\Psi} = dD_q^{\phi}$$

- Correlation dimension
  - Kravtsov and Cuevas '07

$$\langle |\Psi_n(\mathbf{r})|^2 |\Psi_m(\mathbf{r})|^2 \rangle \propto |E_n - E_m|^{-[1-(D_2/d)]}$$

Correlations at the transition

Cross-correlations of multifractal eigenstates

$$
\begin{aligned}
C_{nm} &= L^d \int d^d r \left\langle |\psi_n(\mathbf{r})|^2 |\psi_m(\mathbf{r})|^2 \right\rangle \\
&= \begin{cases} (E_c/[\max\{|\omega_{nm}|, \Delta\}])^\gamma, & 0 < |\omega_{nm}| < E_c, \\ (E_c/|\omega_{nm}|)^2, & E_c < |\omega_{nm}| \end{cases}
\end{aligned}
$$

Auto-correlation of multifractal eigenstates

$$
\begin{aligned}
C_{\mathbf{r}_1,\mathbf{r}_2} &= L^{2d} \left\langle |\psi_n(\mathbf{r}_1)|^2 |\psi_n(\mathbf{r}_2)|^2 \right\rangle_n \\
&= \left(\frac{L}{|\mathbf{r}_1 - \mathbf{r}_2|} \right)^{d\gamma}, \qquad 0 < |\mathbf{r}_1 - \mathbf{r}_2| < L
\end{aligned}
$$

where

$$\gamma = 2(\alpha_0 - d)/d = (1 - d_2/d)$$

Scattering: system + lead

- Scattering matrix

$$S(E) = e^{i\Phi(E)} = 1 - 2i\pi\, W^T\, (E\mathbf{1} - \mathcal{H}_{\text{eff}})^{-1}\, W$$

$$\mathcal{H}_{\text{eff}} = H - i\pi W W^T \qquad W_n = w_0 \delta_{nn_0}$$

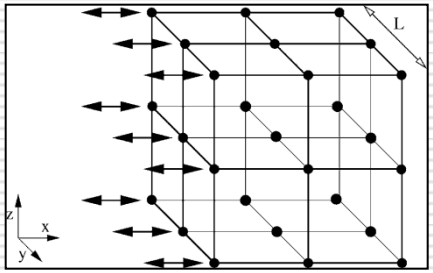
- Wigner-Smith delay time

$$\tau(E=0) = \left.\frac{d\Phi(E)}{dE}\right|_{E=0} = -2\mathrm{Im}\mathrm{Tr}(E - \mathcal{H}_{\text{eff}})^{-1}\big|_{E=0}$$

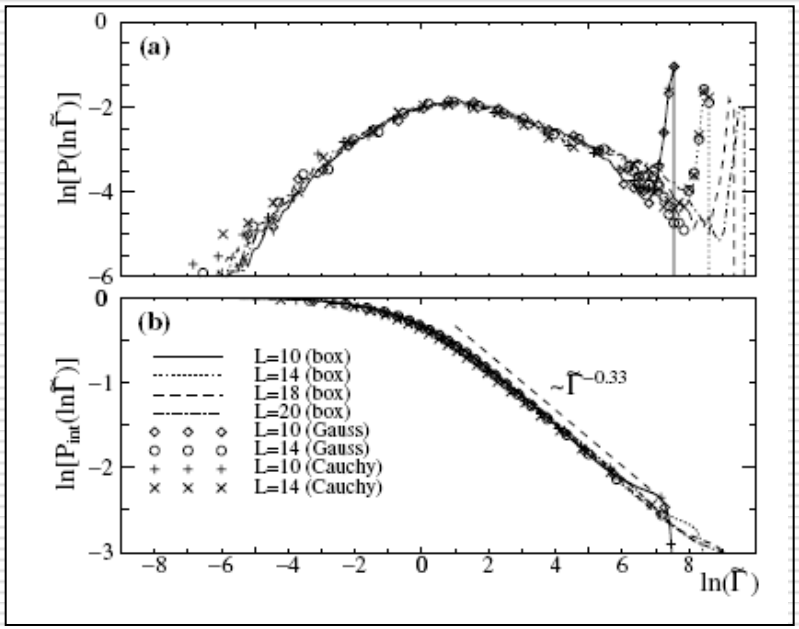
- Rezonance widths: $\mathcal{E}_n = E_n - i\Gamma_n/2$

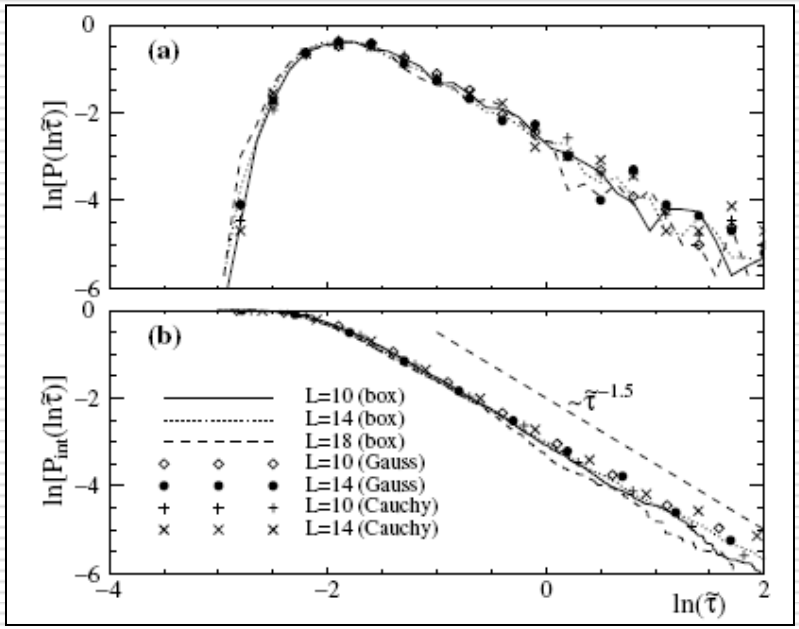
  eigenvalues of $\mathcal{H}_{\text{eff}}$ poles of $S(E)$

Scattering, open system



$$\mathcal{P}(\tilde{\Gamma}) \sim \tilde{\Gamma}^{-(1+1/3)}, \qquad \tilde{\Gamma} \equiv \Gamma/\Delta \qquad \mathcal{P}(\tilde{\tau}) \sim \tilde{\tau}^{-2.5}, \qquad \tilde{\tau} \equiv \tau M \Delta$$





Find the transition with a stopwatch!

Kottos and Weiss ’02; Weiss, *et al.* ‘06

Effect of multifractality

- Scattering
- Conductance, shot-noise
- LDOS fluctuations, dynamics
- Entanglement
- Magnetic impurities

Effect of multifractality

Generalize!

⇩

Take the model of the model!

⇩

PBRM

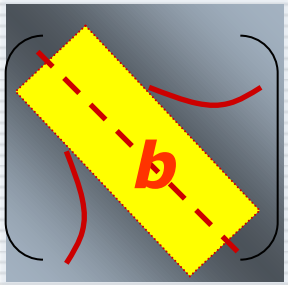
PBRM: Power-law Band Random Matrix

- model: $N \times N$ matrix, $\langle H_{ij} \rangle = 0$

$$\langle (H_{ij})^2 \rangle = \frac{1}{\beta} \left(1 + \left[\frac{\sin(\frac{\pi}{N}|i-j|)}{\frac{\pi}{N} b} \right]^{2\mu} \right)^{-1}$$

- asymptotically: $(1 \ll |i-j| \ll N-1)$

$$\langle (H_{ij})^2 \rangle \sim \left(\frac{b}{|i-j|} \right)^{2\mu}$$

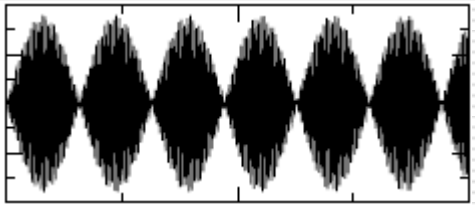


- free parameters μ and b

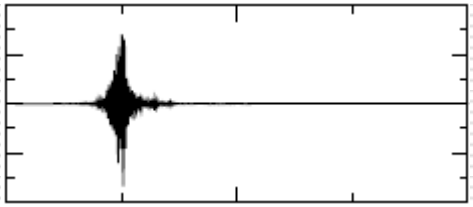
Mirlin, *et al.* '96, Mirlin '00

PBRM

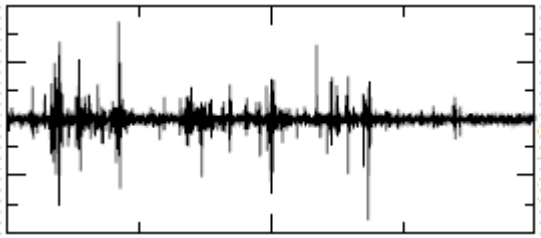
- for $\mu < 1$ $\Rightarrow$ RMT, as if $\mu \to 0$
  - for $1/2 < \alpha < 1 \Rightarrow$ similar to metal with $d = 1/(\mu - 1/2)$



- for $\mu > 1$ $\Rightarrow$ BRM $\Rightarrow$ Poisson, as if $\mu \to \infty$
  - for $\mu > 3/2 \Rightarrow$ power law localization with exponent μ (cf. Yeung-Oono '87)

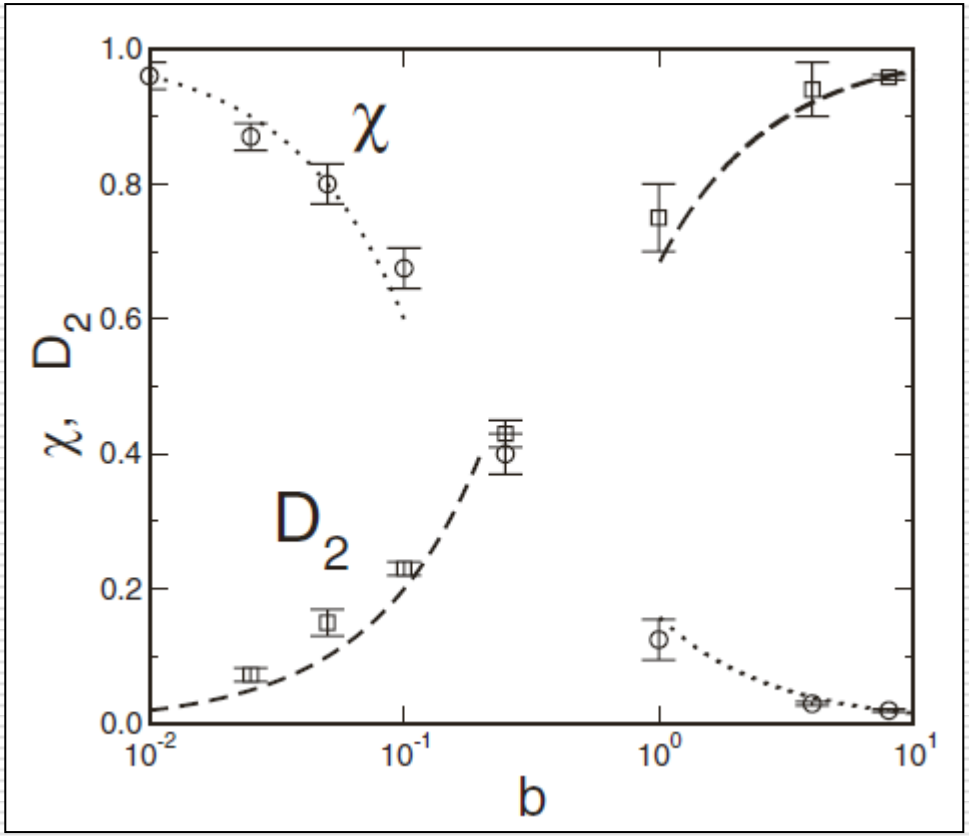


- for $\mu = 1$ $\Rightarrow$ criticality (cf. Levitov '90)
  - continuous line of transitions: b



Mirlin, *et al.* '96, Mirlin '00

PBRM at criticality ($\mu = 1$)



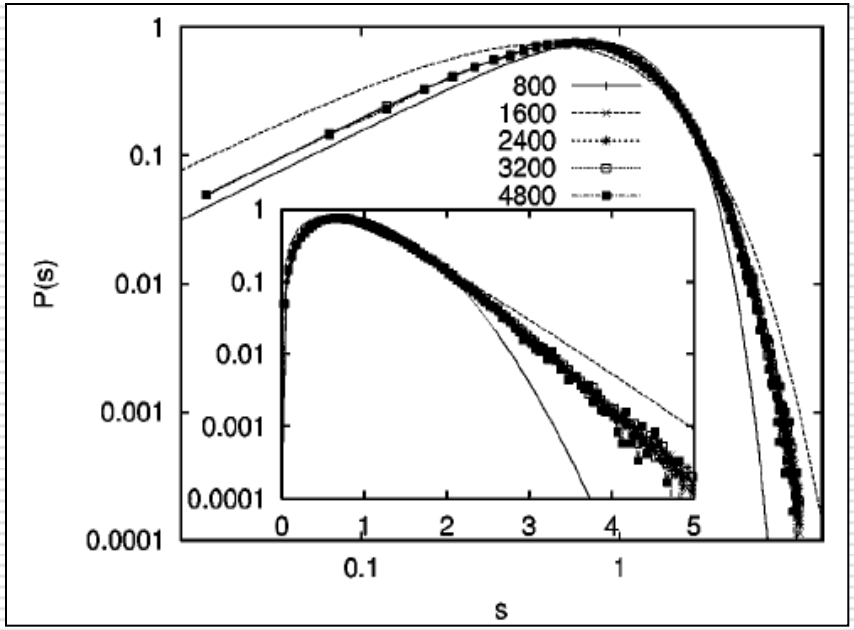
- $b \gg 1$ weak multifractality $D_2 \longrightarrow 1 \quad \chi \longrightarrow 0$
- $b \ll 1$ strong multifractality $D_2 \longrightarrow 0 \quad \chi \longrightarrow 1$

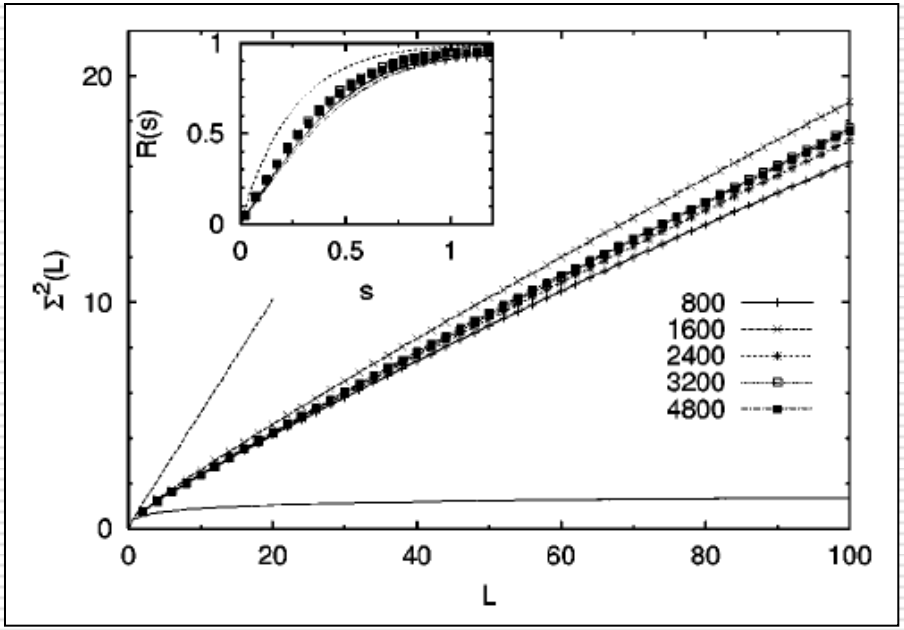
$$\chi \sim \left(1 + a_\chi b\right)^{-1}$$

Mirlin, *et al.* '96, Mirlin '00

PBRM at criticality ($b = 1$)

semi-Poisson statistics qualitatively valid

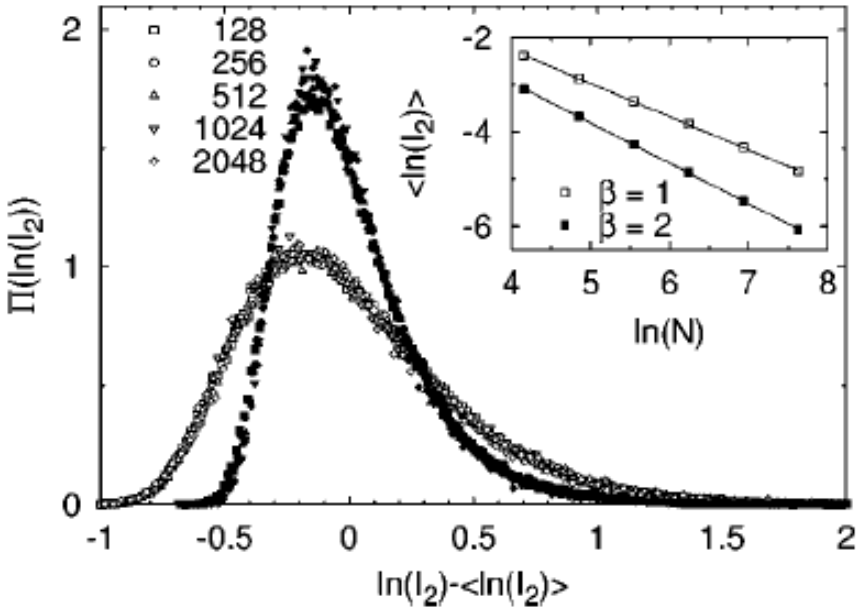


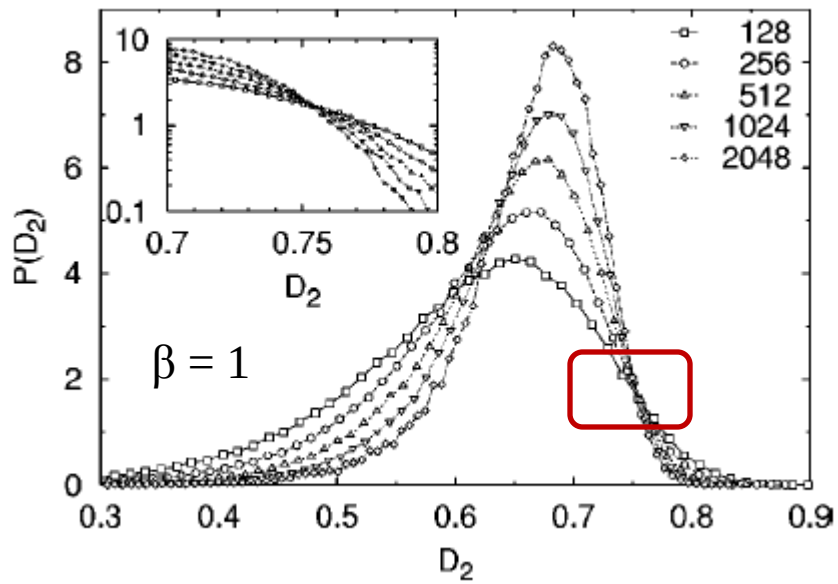


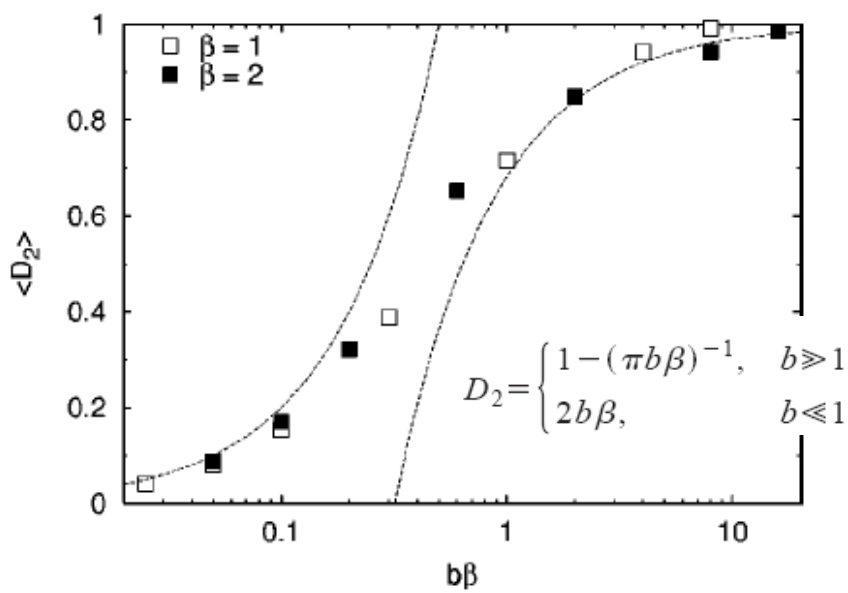
IV and Braun (2000)

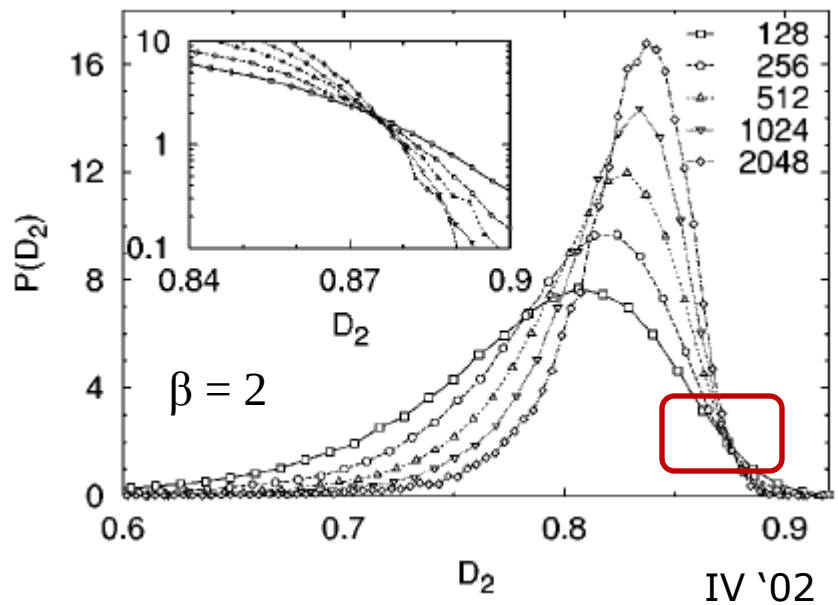
joint distribution

state-to-state fluctuation



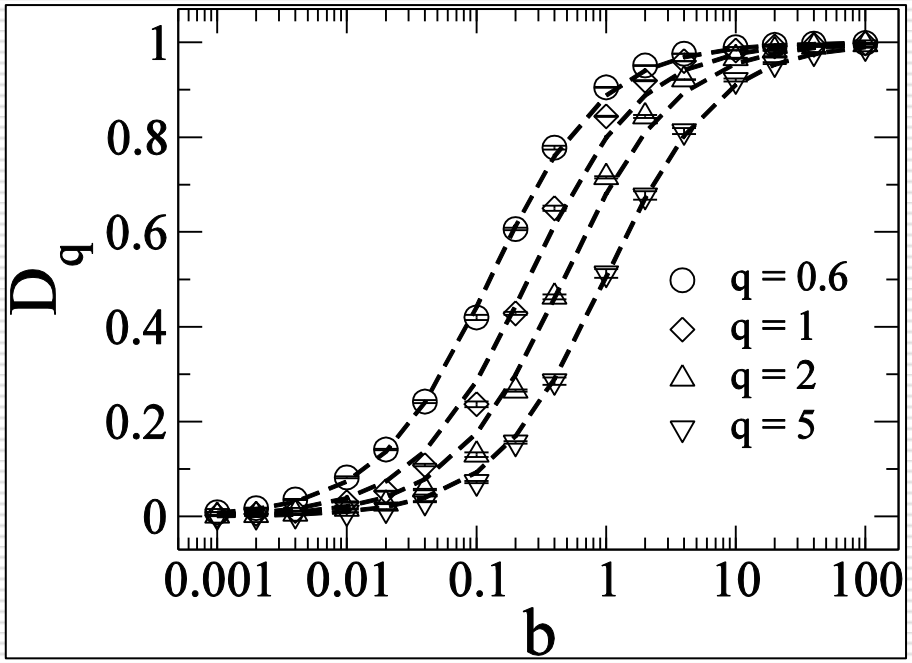






IV '02

Generalized dimensions vs. *b*



$$D_q \approx \left[1 + \left(a_q b\right)^{-1}\right]^{-1}$$

$$D_{q'} \approx \frac{q D_q}{q' + (q - q') D_q}$$

JAMB és IV (2012)

PBRM at criticality $(0.5 < q < 5)$

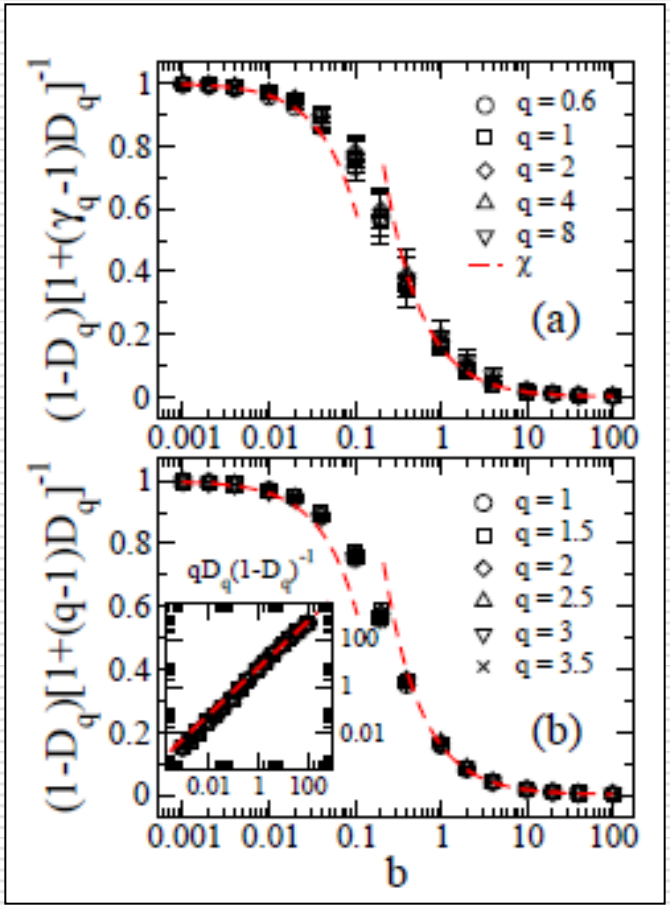
General relations

$$\frac{qD_q}{1-D_q} \approx \text{const} \sim b$$

Spectral statistics and D_q

$$\chi \approx \frac{1-D_q}{1+(q-1)D_q}$$

e.g.: $$D_2 \approx \frac{D_1}{2-D_1} \approx \frac{1-\chi}{1+\chi}$$



JAMB és IV (2012, 2013)

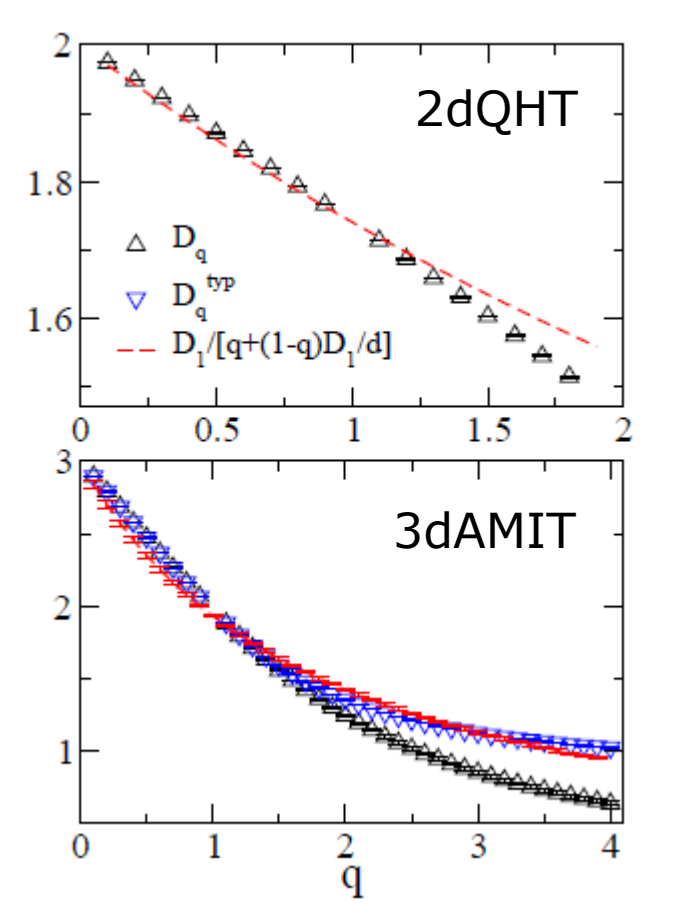
Higher dimensions, $q < 0.5$

Replace $$D_q \to \frac{D_q}{d}$$

For $q < 0.5$ using $\Delta_q = \Delta_{1-q}$

$$D_q \approx \frac{1-2q}{1-q} + \frac{q}{1-q}\left(\frac{D_1}{1+q(D_1-1)}\right)$$

$$\chi \approx \frac{1-D_q}{q(2-D_q)}$$



JAMB és IV (2013)

Scattering: PBRM + 1 lead

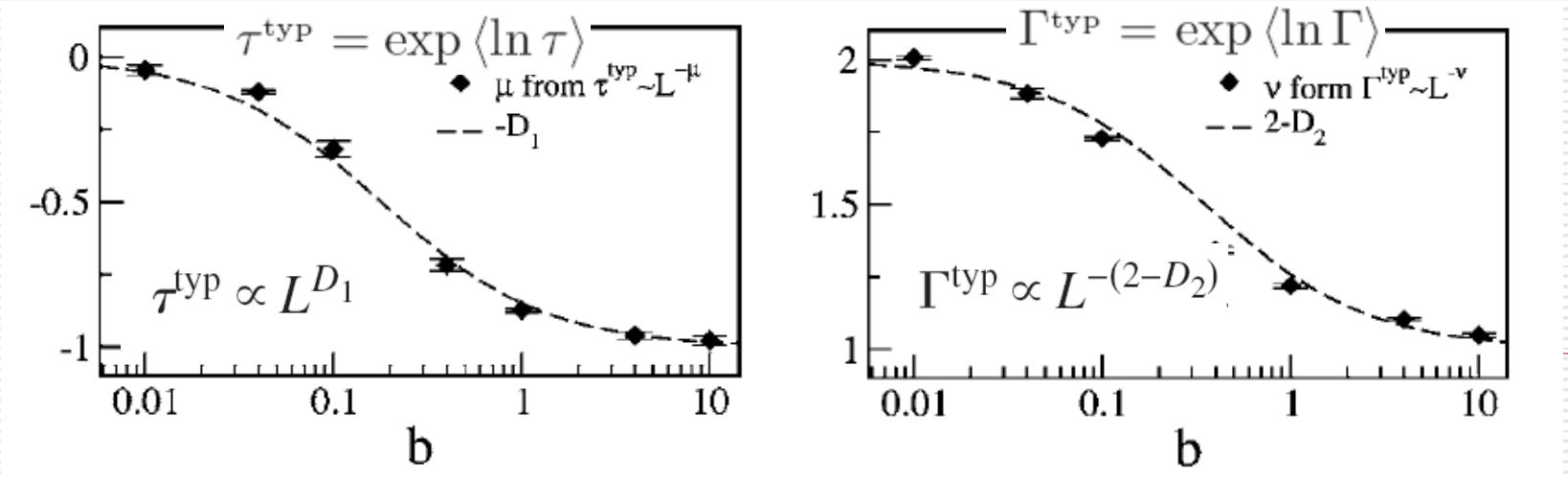
- JA Méndez-Bermúdez – Kottos '05
  Ossipov – Fyodorov '05:

$$\langle \tau^{-q} \rangle \propto L^{-qD_{q+1}}$$

$$\tilde{\mathcal{P}}_w(\tilde{\tau}_w) = \frac{1}{\tilde{\tau}_w^3} \mathcal{P}_y\left(\frac{1}{\tilde{\tau}_w}\right)$$

Measure the multifraclity using a stopwatch!

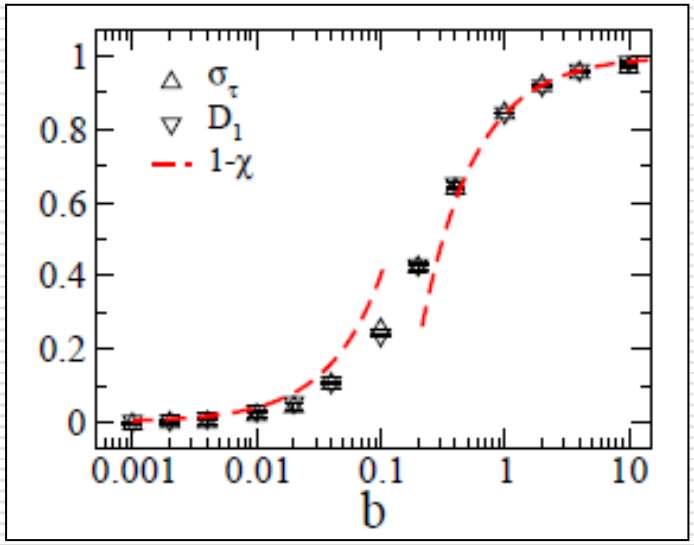
- JA Méndez-Bermúdez – IV 06:



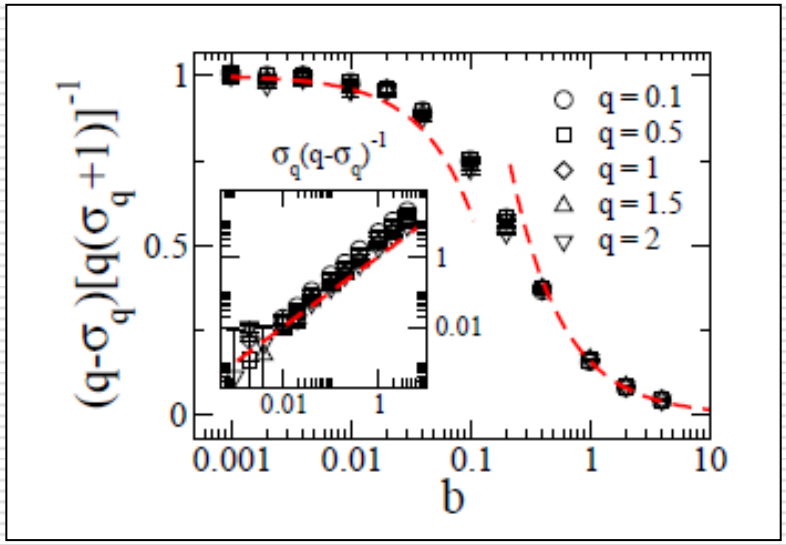
Scattering exponents

Wigner-Smith delay time

$$\exp\langle \ln \tau \rangle \sim L^{\sigma_\tau}$$



$$\langle \tau^{-q} \rangle \sim L^{-\sigma_q}$$



$$\sigma_\tau = 1 - \chi$$

$$\sigma_q = \frac{q(1-\chi)}{1+q\chi}$$

JAMB és IV (2013)

PRL 103, 126401 (2009) PHYSICAL REVIEW LETTERS week ending 18 SEPTEMBER 2009

Critical Metal Phase at the Anderson Metal-Insulator Transition with Kondo Impurities

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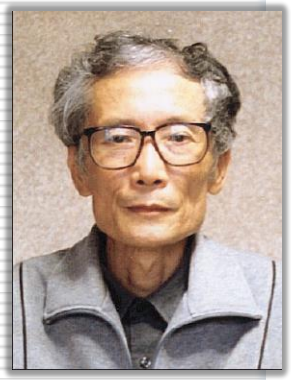
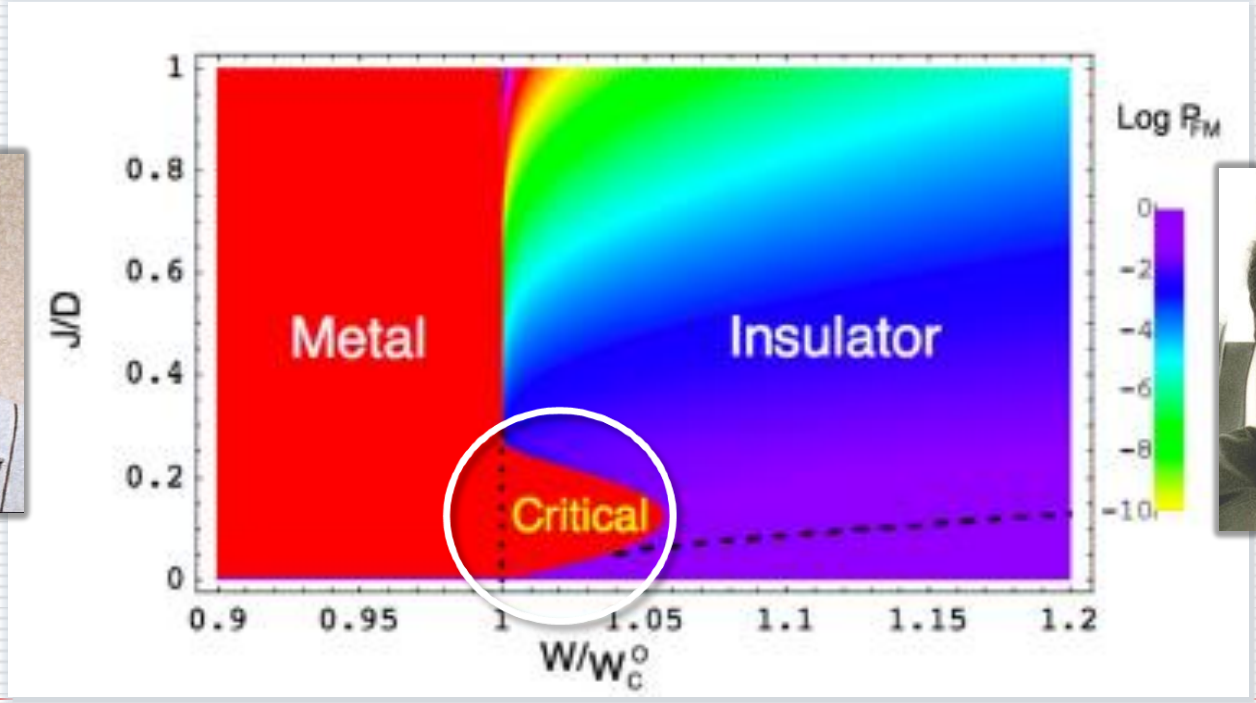
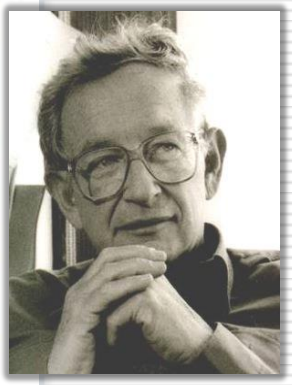
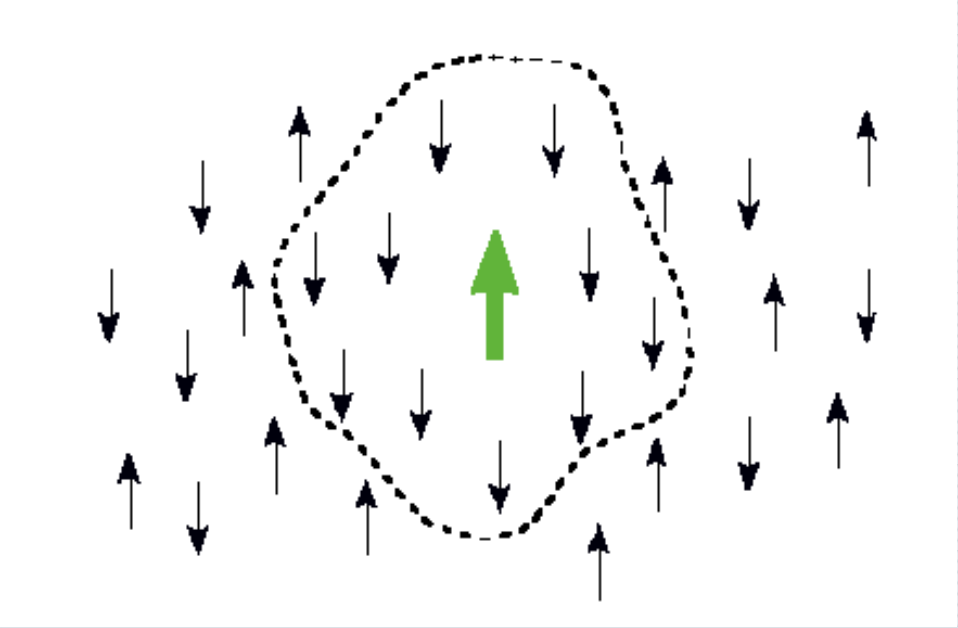


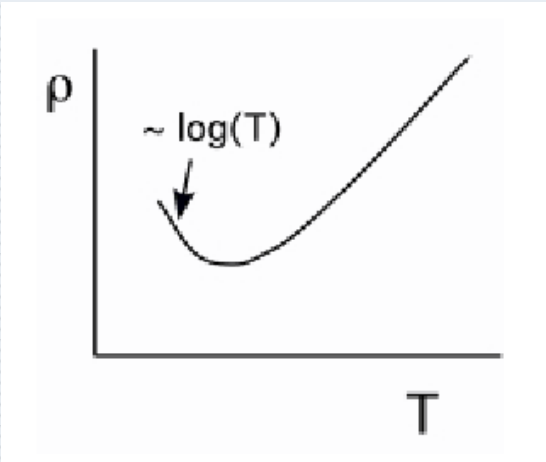




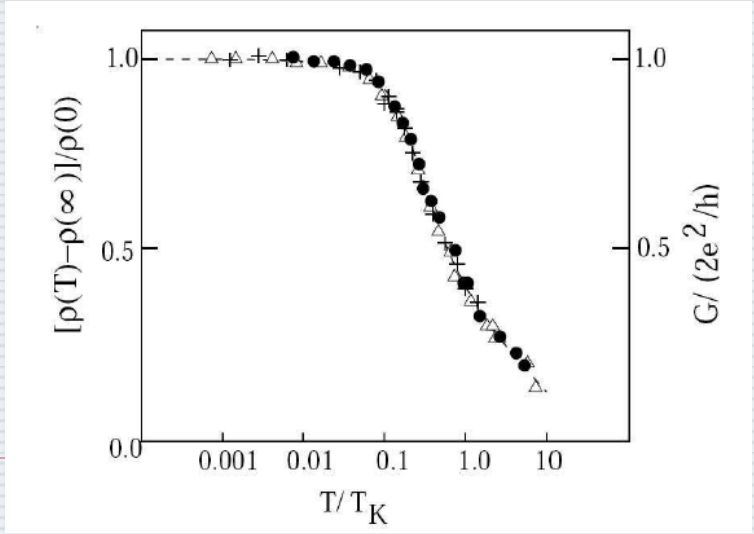
Kettemann, Mucciolo, IV, Slevin (2012)

Kondo effect in metals (1964)

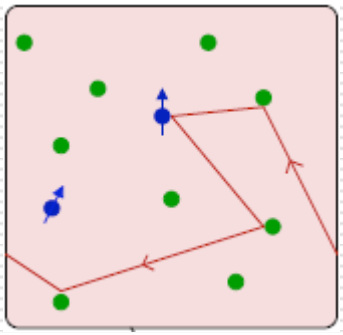




below $T < T_K$ spin-flip scattering
singlet ground-state
Kondo-screening



Kondo effect in disordered metals



Metal

Weak disorder → extended states
Full screening, **no** free magnetic moments

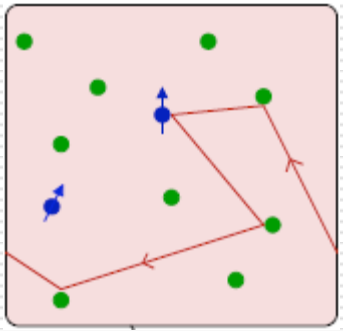
Insulator

Strong disorder → localized states,
No screening → free magnetic moments exist

Criticality → multifractality

Screening? Frozen Kondo-effect?
Are there any free magnetic moments?

Kondo effect in disordered metals



Local T_K → broad, bimodal $P(T_K)$

$$H = H_0 + J\,\vec{S}\cdot\vec{s}(0)$$

1-loop (Nagaoka – Suhl):

$$\frac{1}{J} = \frac{1}{2N}\sum_{n=1}^{N}\frac{L^d|\psi_n(0)|^2}{E_n - E_F}\tanh\left(\frac{E_n - E_F}{2T_K}\right)$$

Unscreened (free) magnetic moments exist:

$T_K = 0$ ha $J < J_c$

$$\frac{1}{J_c} = \frac{1}{2N}\sum_{n=1}^{N}\frac{L^d|\psi_n(0)|^2}{|E_n - E_F|}$$

Weekly disordered conductor:

$$J_c \sim D/\ln N \to 0$$

D is the bandwidth

Insulator:

$$J_c \sim D/\ln N_I, \quad N_I = D/\Delta_I$$
$$\Delta_I = (\nu\,\xi^d)^{-1}$$

Kondo effect at the critical point

log-normally distributed wave functions

$$P(\alpha_l) \sim L^{-d+f(\alpha_l)}$$

$$\alpha_l = -\frac{\ln |\psi_l(0)|^2}{\ln L}$$

$$f(\alpha) = d - \frac{(\alpha - \alpha_0)^2}{4(\alpha_0 - d)}$$

joint distribution of wave function intensities

$$P_\alpha(\alpha_l, \alpha_k) = L^{a_{lk}[f(\alpha_l)+f(\alpha_k)-2d]+b_{lk}\frac{(\alpha_l-\alpha_0)(\alpha_k-\alpha_0)}{2(\alpha_0-d)}}$$

$$a_{kl} = \left(1 + \sqrt{1 + 4b_{kl}^2}\right)/2 \qquad b_{kl} = g_{kl}/(g_{kl}^2 - 1)$$

energy correlations of eigenstates

$$g_{lk} = \frac{\ln(|\omega_{lk}|/E_c)}{\ln L} \times \begin{cases} d^{-1}, & |\omega_{kl}| < E_c, \\ (\alpha_0 - d)^{-1}, & |\omega_{lk}| > E_c. \end{cases}$$

$$\omega_{lk} = E_l - E_k$$

$$\frac{J_c(\alpha)}{D} = \frac{\alpha - \alpha_0}{d}\left\{1 - \exp\left[-\frac{(\alpha - \alpha_0)\sqrt{\ln L}}{2\sqrt{\alpha_0 - d}}\right]\right\}^{-1}$$

Kondo effect arounc the critical point

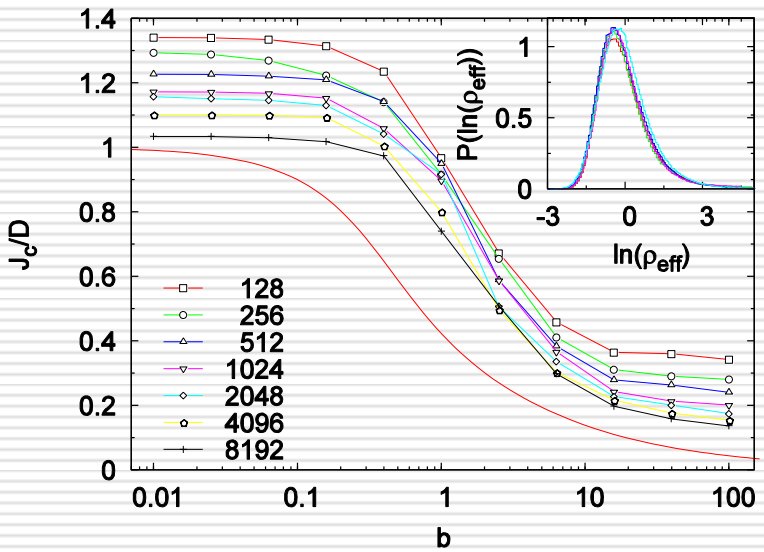
$$P_{\mathrm{FM}} = n_{\mathrm{FM}}/n_M = \mathrm{Erfc}\left(\sqrt{\frac{\ln\xi}{2\gamma}}\frac{J}{D}\right)$$

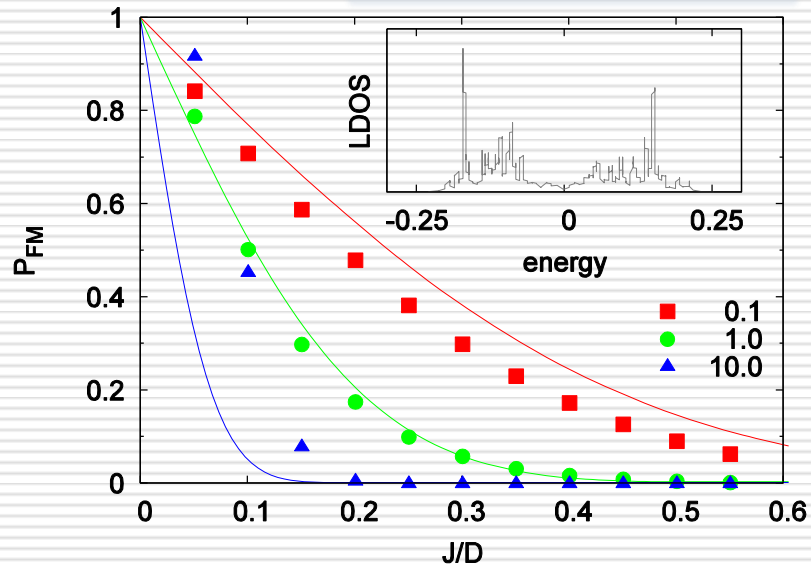
$$\gamma = 2(\alpha_0 - d)/d$$

Free magnetic moments away from the critical point

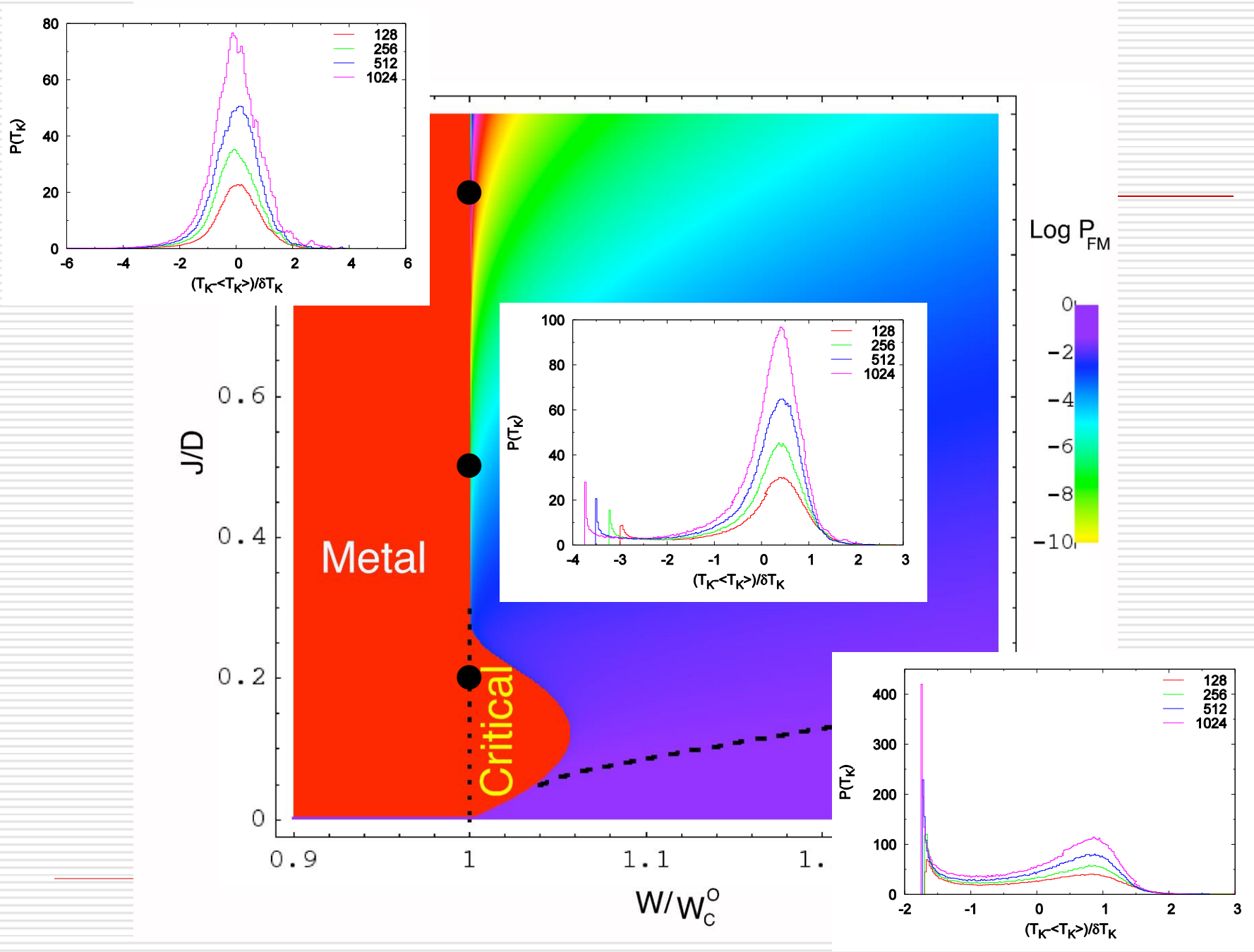
$$P_{\mathrm{FM}} \sim (W - W_c)^{\kappa(J)}$$

$$\kappa(J) = (\nu d/2\gamma)(J/D)^2$$





$$J_c^{(1)} = \sqrt{2\gamma}D \longrightarrow n_{\mathrm{FM}} = n_M/N$$


Log P_{FM}
0
-2
-4
-6
-8
-10
J/D
0.6
0.4
0.2
0
0.9
1
1.1
1.
W/W_c^o
Metal
Critical
$P(T_K)$
$(T_K-\langle T_K\rangle)/\delta T_K$
128
256
512
1024

Symmetry class of the transition at T=0K

$$X_s = \xi^2/\mathcal{D}\tau_s$$

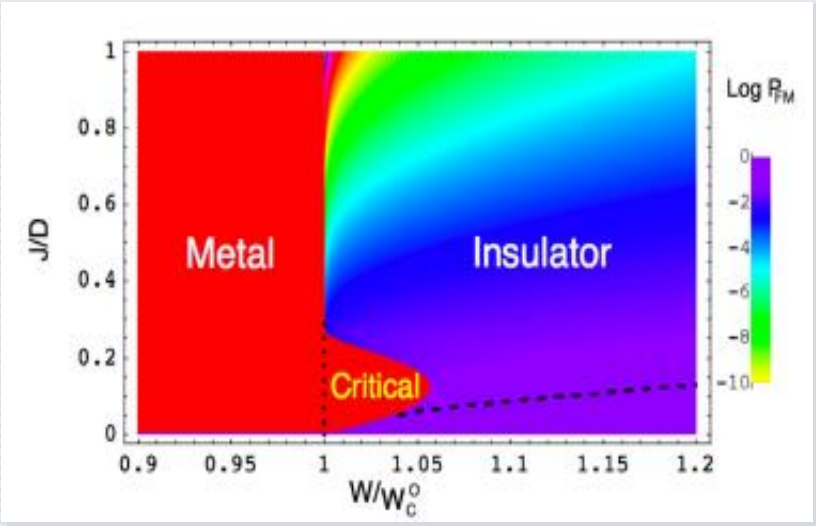
Symmetry-dependent critical point $W_c^{\mathrm{orth}} < W_c^{\mathrm{unit}}$

In the case if $J < J^* = \left(2\sqrt{\gamma/d} - \gamma\right) D$ and intermediate phase: $X_s = 1$

Since $$\frac{1}{\tau_{FM}} = \frac{1}{\tau_s^0}\left(\frac{\xi}{a_c}\right)^{-\frac{d}{2\gamma}\left(\frac{J}{D}+\gamma\right)^2}$$

hence $$W_c(J) = W_c^O\left[1 + \left(\frac{a_c}{D_c\tau_s^0}\right)^{\eta(J)}\right]$$

where $$\eta(J) = \left[2\nu - \frac{d}{2\gamma}\left(\frac{J}{D}+\gamma\right)^2\right]^{-1}$$



Summary and outlook

- Az Anderson-modell és MIT
  - New experimental evidence
  - New topics involved
- Multifractal states in general
  - Random matrix models (PBRM)
- Outlook
  - Interacting particles (Coulomb) (cf. Mirlin, Kravtsov)
  - Decoherence – robust fidelity
  - Proximity (SC) – subgap spectrum
  - Topological insulators – ?
  - QCD ??????