NNLO QCD calculations at work: determining $\alpha_s$ in $e^+e^-$ annihilation

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Introduction
• *$\alpha_S$* is one fundamental parameter of the standard model

$\implies$ its most precise determination is mandatory

- World average: $\alpha_S(M_Z) = 0.1181 \pm 0.0011$
- World average from various observables: from lattice to hadron collisions
- Most of the measurements were done in $e^+e^-$
- $e^+e^-$ is still not fully exploited:
  - new observables
  - new calculations
  - new tools to estimate non-perturbative effects
- LHC measurements are just coming up…
**Introduction**

- $\alpha_s$ determination not only needs precise measurements but theoretical predictions with equally high quality.

- The new $\alpha_s$ value has to meet certain criteria to end up contributing to the world average:

  1. We need at least Next-to-next-to-leading order (NNLO) accuracy in the theoretical calculation.

    
    *Luckily this can be fulfilled…*

2014 Review of Particle Properties [2]. In the following, those new results which are used to determine the new world average value of $\alpha_s$, i.e. those that are based on at least complete next-to-next-to-leading order (NNLO) perturbation theory, are published in peer-reviewed journals and contain complete estimates of experimental and systematic uncertainties, will be summarised. Also results which are used for demonstrating asymptotic freedom, i.e. the specific energy dependence of $\alpha_s$ as predicted by Quantum Chromodynamics, even if being based on next-to-leading (NLO) perturbation theory only, will be reviewed.

The newest and most actual entries satisfying the quality criteria given above are:

- updated results from $\tau$-decays [4] [5] [6], based on a re-analysis of ALEPH data and on complete N$^3$LO perturbation theory,
- more results from unquenched lattice calculations, [7][8],
- further results from world data on structure functions, in NNLO QCD [9],
- from $e^+e^-$ annihilation, hadronic event shapes (C-parameter) in soft collinear effective field theory matched to NNLO perturbation theory [10],
- $\alpha_s$ determinations at LHC, from data on the ratio of inclusive 3-jet to 2-jet cross sections [11], from inclusive jet production [12], from the 3-jet differential cross section [13], and from energy-correlations [14], all in NLO QCD, plus one determination in complete NNLO, from a measurement of the $t\bar{t}$ cross section at $\sqrt{s} = 7$ TeV [15];
- and finally, an update of $\alpha_s$ from a global fit to electroweak precision data [16], based on complete N$^3$LO perturbation theory.
Nuts and bolts of QCD perturbation theory
• We are always interested in cross sections, e.g. for a jet function $J$.
• Treat $\alpha_S$ as the perturbation parameter, thus expand in powers of $\alpha_S$:

$$\sigma[J] = \sigma^{\text{LO}}[J] + \sigma^{\text{NLO}}[J] + \sigma^{\text{NNLO}}[J] + \ldots$$

• To be calculable $J$ has to be IR safe/finite:
  arbitrary number of soft and/or collinear partons should not change the value of $J$.
• Beyond LO singularities appear in each order in different contributions but vanish when contributions are combined due to the KLN theorem.

$$\sigma^{\text{NLO}}[J] = \sigma^{\text{NLO}}_{m+1}[J] + \sigma^{\text{NLO}}_m[J] = \int_{m+1} d\sigma^R_{m+1} J_{m+1} + \int_m d\sigma^V_m J_m$$

one more final state parton (kinematic singularity)
cancellation
one more loop ($\epsilon$ poles)
QCD perturbation theory

• There would be no problem if we could calculate everything analytically.
• This is not possible, need something numerically integrable (with Monte-Carlo techniques).
• One possible remedy is subtraction.

\[ I = \lim_{\epsilon \to 0} \left( \int_0^1 \frac{dx}{x^{1-\epsilon}} f(x) - \frac{1}{\epsilon} f(0) \right) \]

• As \( \epsilon \to 0 \) the sum is finite, terms are separately divergent.
• Find a \( g(x) \) function which mimics \( f(x) \) as \( x \to 0 \):

\[ I = \lim_{\epsilon \to 0} \int_0^1 \frac{dx}{x^{1-\epsilon}} [f(x) - g(x)] + \lim_{\epsilon \to 0} \left( \int_0^1 \frac{dx}{x^{1-\epsilon}} g(x) - \frac{1}{\epsilon} f(0) \right) \]

• \( g(x) \) has to be analytically integrable…

subtraction

integrated subtraction
At NLO the subtraction looks like:

\[ \sigma^{NLO}[J] = \sigma^{NLO}_{m+1}[J] + \sigma^{NLO}_m[J] \]

\[ \sigma^{NLO}_{m+1}[J] = \int_{m+1} d\sigma^R_{m+1} J_{m+1} \]

\[ \sigma^{NLO}_m[J] = \int_m d\sigma^V_m J_m \]
QCD perturbation theory

• At NLO the subtraction looks like:

\[ \sigma^{\text{NLO}}[J] = \sigma^{\text{NLO}}_{m+1}[J] + \sigma^{\text{NLO}}_m[J] \]

\[ \sigma^{\text{NLO}}_{m+1}[J] = \int_{m+1} \left[ d\sigma^{\text{R}}_{m+1} J_{m+1} - d\sigma^{\text{R,A}}_{m+1} J_m \right]_{d=4} \]

\[ \sigma^{\text{NLO}}_m[J] = \int_m d\sigma^V_m J_m \]

• Kinematic singularities emerging in single parton emissions in the \( m+1 \) parton contribution regularized by the subtractions
QCD perturbation theory

- At NLO the subtraction looks like:

\[ \sigma^{\text{NLO}}[J] = \sigma^{\text{NLO}}_{m+1}[J] + \sigma^{\text{NLO}}_m[J] \]

- \( \sigma^{\text{NLO}}_{m+1}[J] = \int_{m+1} d\sigma^{R,m+1}_{m+1} J_{m+1} - d\sigma^{R,A,m+1}_{m+1} J_m \) \( d=4 \)

- \( \sigma^{\text{NLO}}_m[J] = \int_m d\sigma^{V}_m J_m + \int_1 d\sigma^{R,A}_m J_m \) \( d=4 \)

- Kinematic singularities emerging in single parton emissions in the m+1 parton contribution regularized by the subtractions

- Explicit \( \epsilon \) poles coming from the loop integral are cancelled by those \( \epsilon \) poles arising from the integrated subtraction terms
At NNLO the philosophy is the same but the subtractions are more elaborate:

\[ \sigma^{\text{NNLO}}[J] = \sigma^{\text{NNLO}}_{m+2}[J] + \sigma^{\text{NNLO}}_{m+1}[J] + \sigma^{\text{NNLO}}_m[J] \]

- \[ \sigma^{\text{NNLO}}_{m+2}[J] = \int_{m+2} \! d\sigma_{m+2}^{\text{RR}} J_{m+2} \]

- \[ \sigma^{\text{NNLO}}_{m+1}[J] = \int_{m+1} \! d\sigma_{m+1}^{\text{RV}} J_{m+1} \]

- \[ \sigma^{\text{NNLO}}_m[J] = \int_m \! d\sigma_m^{\text{VV}} J_m \]
At NNLO the philosophy is the same but the subtractions are more elaborate:

\[ \sigma^{\text{NNLO}}[J] = \sigma_{m+2}^{\text{NNLO}}[J] + \sigma_{m+1}^{\text{NNLO}}[J] + \sigma_{m}^{\text{NNLO}}[J] \]

\[ \sigma_{m+2}^{\text{NNLO}}[J] = \int_{m+2} \left\{ d\sigma_{m+2}^{\text{RR}} J_{m+2} - d\sigma_{m+2}^{\text{RR,}A_2} J_{m} - \left[ d\sigma_{m+2}^{\text{RR,}A_1} J_{m+1} - d\sigma_{m+2}^{\text{RR,}A_{12}} J_{m} \right] \right\} d=4 \]

\[ \sigma_{m+1}^{\text{NNLO}}[J] = \int_{m+1} d\sigma_{m+1}^{\text{RV}} J_{m+1} \]

\[ \sigma_{m}^{\text{NNLO}}[J] = \int_{m} d\sigma_{m}^{\text{VV}} J_{m} \]

Kinematic singularities in single and double parton emissions in the m+2 parton contribution regularized by the subtractions.
QCD perturbation theory

• At NNLO the philosophy is the same but the subtractions are more elaborate:

\[ \sigma_{\text{NNLO}}^m[J] = \sigma_{m+2}^{\text{NNLO}}[J] + \sigma_{m+1}^{\text{NNLO}}[J] + \sigma_{m}^{\text{NNLO}}[J] \]

\[ \sigma_{m+2}^{\text{NNLO}}[J] = \int_{m+2} \left\{ d\sigma_{m+2}^{\text{RR}} J_{m+2} - d\sigma_{m+2}^{\text{RR}, A_2} J_m - \left[ d\sigma_{m+2}^{\text{RR}, A_1} J_{m+1} - d\sigma_{m+2}^{\text{RR}, A_1 A_2} J_m \right] \right\} d=4 \]

\[ \sigma_{m+1}^{\text{NNLO}}[J] = \int_{m+1} \left\{ \left( d\sigma_{m+1}^{\text{RV}} + \int_1 d\sigma_{m+2}^{\text{RR}, A_1} \right) J_{m+1} - \left[ d\sigma_{m+1}^{\text{RV}, A_1} + \left( \int_1 d\sigma_{m+2}^{\text{RR}, A_1} \right)^{A_1} \right] J_m \right\} d=4 \]

\[ \sigma_{m}^{\text{NNLO}}[J] = \int_m d\sigma_{\text{VV}} [J_m] \]

• Kinematic singularities in single and double parton emissions in the \(m+2\) parton contribution regularized by the subtractions

• Kinematic singularities due to single emissions and pole cancellations with integrated subtraction terms in the \(m+1\) parton line
At NNLO the philosophy is the same but the subtractions are more elaborate:

\[
\sigma^{\text{NNLO}}[J] = \sigma^{\text{NNLO}}_{m+2}[J] + \sigma^{\text{NNLO}}_{m+1}[J] + \sigma^{\text{NNLO}}_m[J]
\]

- Kinematic singularities in single and double parton emissions in the \( m+2 \) parton contribution regularized by the subtractions.
- Kinematic singularities due to single emissions and pole cancellations with integrated subtraction terms in the \( m+1 \) parton line.
- Explicit poles cancelled across the two-loop and integrated subtraction terms.
• The philosophy of local subtraction was used in the construction of the CoLoRFuLNNLO method (Del Duca, Somogyi and Trocsanyi)

• It is derived from first principles: factorization and limit behavior of QCD amplitudes

• It defines fully local subtractions: beneficial for good numerical properties

• Derived for colorless initial states (electron-positron annihilation)

• The generalization for colored initial states (hadron-hadron [LHC] and hadron-lepton [DIS] collisions) is on the way

• The theory was well laid down but needed a powerful numerical framework…
MCCSM
MCCSM stands for Monte-Carlo for the CoLoRFuL Subtraction Method

- Fortran90 program
- Fully automatic and general capable of calculating the NNLO QCD corrections for any process at electron-positron annihilation
- More than 80k lines of code excluding the matrix elements
- Flexible, user friendly and equipped with many features
- Tested in 2- and 3-jet production
- Already used to obtain new observables:
  - Oblateness and Energy-energy correlation
    [Del Duca, Duhr, AK, Somogyi et al. PRL 117 (2016) no.15, 152004]
  - Jet-cone energy fraction
    [Del Duca, Duhr, AK, Somogyi et al. PRD94 (2016) no.7, 074019]
MCCSM

Generating subprocesses

Number of subprocesses at the NNLO-level:

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We found the following relations for:

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<td>$e^+ e^- \rightarrow b \ b^- c^+ c^- g$</td>
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MCCSM

- Generating subprocesses
- Find numerical relations
- Detecting singular regions

----- Cirs -----

item: 1, b (3) -> b (3) || g (5) |
UBorn: e+ e- -> b b- g
     \-> b g g

item: 2, b (3) -> b (3) || g (5) |
UBorn: e+ e- -> b b- g
     \-> b g g

item: 3, b (3) -> b (3) || g (6) |
UBorn: e+ e- -> b b- g
     \-> b g g

item: 4, b~(4) -> b~(4) || g (5) |
UBorn: e+ e- -> b b- g
     \-> b~ g g

item: 5, b~(4) -> b~(4) || g (5) |
UBorn: e+ e- -> b b- g
     \-> b~ g g

item: 6, b~(4) -> b~(4) || g (6) |
UBorn: e+ e- -> b b- g
MCCSM

- Generating subprocesses
- Find numerical relations
- Detecting singular regions
- Checking limiting behavior

CSirs: g (6) -> g (6) | | g (7), g (5) -> 0  VALID
iter no. 1 scale no. 1 1.062666349487440613103691
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Resummation 101
Resummation

There are regions of phase space where perturbation theory breaks down:

\[ \alpha_s \ll 1 \] but logs can become large

\[ \alpha_s \log(...) \approx 1 \implies \text{all orders become important in the perturbative series!} \]

Indeed, the LO is:

\[
\frac{1}{\sigma} \frac{d\sigma}{d\tau} \approx 0 \quad \frac{\alpha_S}{2\pi} \frac{4}{C_F} \frac{1}{\tau} \log \tau
\]

This is clearly divergent, to have a meaningful prediction in the small \( \tau \) region have to sum up all orders!

\( \tau=1 \)-Thrust distribution in \( e^+e^- \rightarrow 3 \) jets
Let’s stick to the 1-Thrust distribution, its perturbative expansion:

\[ \frac{d\sigma}{d\tau} \sim \alpha_S \{ A + \alpha_S B + \alpha_S^2 C + \ldots \} \]

Keeping only singular terms order by order:

\[ \frac{d\sigma}{d\tau} \sim \frac{1}{\tau} \sum_{n=1}^{\infty} \sum_{m=0}^{2n-1} \alpha_S^n \log^m \tau \]

or writing out explicitly order by order and grouping accordingly:

\[ \frac{d\sigma}{d\tau} \sim \frac{1}{\tau} \left\{ \alpha_S \left( \log \tau + 1 \right) + \right. \]

\[ + \alpha_S^2 \left( \log^3 \tau + \log^2 \tau + \log \tau + 1 \right) + \]

\[ + \alpha_S^3 \left( \log^5 \tau + \log^4 \tau + \log^3 \tau + \log^2 \tau + \log \tau + 1 \right) + \]

\[ + \ldots \left\} \right. \]
Fixed order calculation:
\[
\frac{d\sigma}{d\tau} \sim \frac{1}{\tau} \left\{ \alpha_S \left( \log \tau + 1 \right) + \alpha_S^2 \left( \log^3 \tau + \log^2 \tau + \log \tau + 1 \right) + \alpha_S^3 \left( \log^5 \tau + \log^4 \tau + \log^3 \tau + \log^2 \tau + \log \tau + 1 \right) + \ldots \right\}
\]

Resummation:
\[
\frac{d\sigma}{d\tau} \sim \frac{1}{\tau} \left\{ \alpha_S \left( \log \tau + 1 \right) + \alpha_S^2 \left( \log^3 \tau + \log^2 \tau + \log \tau + 1 \right) + \alpha_S^3 \left( \log^5 \tau + \log^4 \tau + \log^3 \tau + \log^2 \tau + \log \tau + 1 \right) + \ldots \right\}
\]
Formally the resumed formula for an observable $y$:

$$\frac{1}{\sigma} \frac{d\sigma}{dy} = H(\alpha_S(\mu)) \int_0^\infty dbb J_0(by) S(Q, b)$$

with:

$$S(Q, b) = \exp \left\{ - \int_{b_0^2/b^2}^{Q^2} \frac{dq^2}{q^2} \left[ A(\alpha_S(q^2)) \log \frac{Q^2}{q^2} + B(\alpha_S(q^2)) \right] \right\}$$

and

$$A(\alpha_S) = \sum_{n=1}^\infty \left( \frac{\alpha_S}{4\pi} \right)^n A^{(n)}, \quad B(\alpha_S) = \sum_{n=1}^\infty \left( \frac{\alpha_S}{4\pi} \right)^n B^{(n)}, \quad H(\alpha_S) = 1 + \sum_{n=1}^\infty \left( \frac{\alpha_S}{4\pi} \right)^n H^{(n)}$$

(Q is the CM energy)

To carry out resummation at a given order have to determine the various $A^{(n)}$, $B^{(n)}$ and $H^{(n)}$ terms such as:

- for LL accuracy: $A^{(1)}$
- for NLL accuracy: $A^{(1)}$, $A^{(2)}$, $B^{(1)}$ (and $H^{(1)}$)
- for NNLL accuracy: $A^{(1)}$, $A^{(2)}$, $A^{(3)}$, $B^{(1)}$, $B^{(2)}$ (and $H^{(1)}$ and $H^{(2)}$)
The resumed result contains **some logs exponentiated providing meaningful prediction in some parts of phase space**

Since it only contains **part of the singular terms** the resumed prediction is not at all accurate in the whole available phase space

$\implies$ when logs are not essential have to use the fixed-order prediction

To have good precision over the whole available phase space we need to match the resumed prediction with the fixed-order one

\[
\frac{1}{\sigma} \frac{d\sigma}{dy} \neq \left[ \frac{1}{\sigma} \frac{d\sigma}{dy} \right]_{\text{res.}} + \left[ \frac{1}{\sigma} \frac{d\sigma}{dy} \right]_{\text{f.o.}}
\]

Even the fixed order contains logs $\implies$ these have to be subtracted:

\[
\frac{1}{\sigma} \frac{d\sigma}{dy} = \left[ \frac{1}{\sigma} \frac{d\sigma}{dy} \right]_{\text{res.}} + \left[ \frac{1}{\sigma} \frac{d\sigma}{dy} \right]_{\text{f.o.}} - \left\{ \left[ \frac{1}{\sigma} \frac{d\sigma}{dy} \right]_{\text{res.}} \right\}_{\text{f.o.}}
\]

This is called the (naive) R matching scheme
• If the resumed prediction is not precise enough (not enough Ns) not all the logs are cancelled in the fixed-order calculation

\[ \Rightarrow \text{non-resumed logs in the fixed order spoil physicality for small } y \text{ values} \]

• To ensure physicality in the widest range use a different prescription:

• Define a cumulative observable:

\[ R(y) = \frac{1}{\sigma} \int_{y_0}^{y} dy' \frac{d\sigma(y')}{dy'} \]

• with the constraint of: \( R(y_{\text{max}}) = 1 \)

• Note also that:

\[ \frac{dR(y)}{dy} = \frac{1}{\sigma} \frac{d\sigma(y)}{dy} \]
The fixed-order expansion of $R$ is:

$$[R(y)]_{f.o.} = 1 + \frac{\alpha_S}{2\pi} \bar{A}(y) + \left( \frac{\alpha_S}{2\pi} \right)^2 \bar{B}(y) + \left( \frac{\alpha_S}{2\pi} \right)^3 \bar{C}(y) + O(\alpha_S^4)$$

The resumed formula takes the form of:

$$[R(y)]_{res.} = \left( 1 + C_1 \alpha_S + C_2 \alpha_S^2 + \ldots \right) \exp \left[ L g_1 (\alpha_S L) + g_2 (\alpha_S L) + \alpha_S g_3 (\alpha_S L) + \ldots \right] + O(\alpha_S y)$$

$L$ stands for the essential log

$$g_n (\alpha_S L) = \sum_{i=1}^{\infty} G_{i,i+2-n} \left( \frac{\alpha_S}{2\pi} \right)^i L^{i+2-n}$$
• Taking the log of both and expanding in terms of $\alpha_S$:

$$\log[R(y)]_{\text{f.o.}} = \frac{\alpha_S}{2\pi} \bar{A}(y) + \left(\frac{\alpha_S}{2\pi}\right)^2 \left[ \bar{B}(y) - \frac{1}{2} \bar{A}^2(y) \right]$$

$$+ \left(\frac{\alpha_S}{2\pi}\right)^3 \left[ \bar{C}(y) - \bar{A}(y)\bar{B}(y) + \frac{1}{3} \bar{A}^3(y) \right] + \mathcal{O}(\alpha_S^4)$$

$$\log[R(y)]_{\text{res.}} = Lg_1(\alpha_S L) + g_2(\alpha_S L) + \alpha_S g_3(\alpha_S L)$$

$$+ \alpha_S C_1 + \alpha_S^2 \left( C_2 - \frac{1}{2} C_1^2 \right) + \alpha_S^3 \left( C_3 - C_1 C_2 + \frac{1}{3} C_1^3 \right) + \mathcal{O}(\alpha_S^4)$$

• The terms coming from non-exponentiated contributions in the resumed prediction can be replaced by those coming from the fixed-order one minus those logs which are available in the resumed prediction.
• Hence the master formula for log-R matching:

\[
\log R(y) = Lg_1(\alpha_S L) + g_2(\alpha_S L) + \alpha_S g_3(\alpha_S L) \\
+ \frac{\alpha_S}{2\pi} \left[ \tilde{A}(y) - G_{11} L - G_{12} L^2 \right] \\
+ \left( \frac{\alpha_S}{2\pi} \right)^2 \left[ \tilde{B}(y) - \frac{1}{2} \tilde{A}^2(y) - G_{21} L - G_{22} L^2 - G_{23} L^3 \right] \\
+ \left( \frac{\alpha_S}{2\pi} \right)^3 \left[ \tilde{C}(y) - \tilde{A}(y) \tilde{B}(y) + \frac{1}{3} \tilde{A}^3(y) - G_{32} L^2 - G_{33} L^3 - G_{34} L^4 \right] \\
+ \mathcal{O}(\alpha_S^4)
\]

In the above only those logs were subtracted which are present at NNLL accuracy.
Energy-energy correlation @ NNLO + NNLL


For $\alpha_s$ we need an observable measured precisely enough and for which accurate prediction can be obtained

<table>
<thead>
<tr>
<th>Observable</th>
<th>Resummation</th>
<th>NNLO</th>
<th>NNLO + N</th>
</tr>
</thead>
<tbody>
<tr>
<td>The “6” event shapes</td>
<td>[Chien et al. ’10], [Abbate et al. ’10], [Becher et al. ’12], [Banfi et al. ’14], [Hoang et al. ’14]…</td>
<td>[Gehrmann-De Ridder et al. ’07], [Weinzierl ’09], [Del Duca et al. ’16]</td>
<td>[Gehrmann et al. ’08]</td>
</tr>
<tr>
<td>Jet rate</td>
<td>[Banfi et al. ’16]</td>
<td>[Gehrmann-De Ridder et al. ’07], [Weinzierl ’09], [Del Duca et al. ’16]</td>
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</tr>
<tr>
<td>Oblateness</td>
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<td></td>
</tr>
<tr>
<td>JCEF</td>
<td></td>
<td>[Del Duca et al. ’16]</td>
<td></td>
</tr>
<tr>
<td>EEC</td>
<td>[de Florian et al. ’04]</td>
<td>[Del Duca et al. ’16]</td>
<td></td>
</tr>
<tr>
<td>…</td>
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<td></td>
<td></td>
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For $\alpha_s$ we need an observable measured precisely enough and for which accurate prediction can be obtained.

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<tr>
<td>shapes</td>
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<tr>
<td>…</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
EEC is the normalized energy-weighted cross section defined in terms of the angle between two particles $i$ and $j$ in the event:

$$\frac{1}{\sigma_t} \frac{d\Sigma(\chi)}{d \cos \chi} \equiv \frac{1}{\sigma_t} \int \sum_{i,j} \frac{E_i E_j}{Q^2} d\sigma_{e^+e^- \rightarrow ij + X} \delta(\cos \chi + \cos \theta_{ij})$$

Back-to-back region corresponds to $\chi = 0$

Available resummation corresponds to the back-to-back region ($\chi \rightarrow 0$).
EEC is the normalized energy-weighted cross section defined in terms of the angle between two particles i and j in the event:

\[
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\]

Back-to-back region corresponds to \( \chi = 0 \)
Available resummation corresponds to the back-to-back region ( \( \chi \rightarrow 0 \) ).
Why EEC?

- In $e^+ e^-$ collisions only global event shapes and jet rates were used for $\alpha_s$ extraction (so far).
- EEC is based upon two-particle correlations.
- At $Q = M_Z$ the high(er) hadronization corr.s may affect differently this type of observable.
• In e⁺ e⁻ collisions only global event shapes and jet rates were used for αₛ extraction (so far).
• EEC is based upon two-particle correlations.
• At Q = M₇ the high(er) hadronization corr.s may affect differently this type of observable.
EEC was measured in several experiments at various energies:

<table>
<thead>
<tr>
<th>Exp.</th>
<th>$\sqrt{s}$ ($\langle \sqrt{s} \rangle$) [GeV]</th>
<th>Events</th>
</tr>
</thead>
<tbody>
<tr>
<td>SLD</td>
<td>91.2(91.2)</td>
<td>6476</td>
</tr>
<tr>
<td>OPAL</td>
<td>91.2(91.2)</td>
<td>336247</td>
</tr>
<tr>
<td>OPAL</td>
<td>91.2(91.2)</td>
<td>128032</td>
</tr>
<tr>
<td>L3</td>
<td>91.2(91.2)</td>
<td>169700</td>
</tr>
<tr>
<td>DELPHI</td>
<td>91.2(91.2)</td>
<td>120600</td>
</tr>
<tr>
<td>TOPAZ</td>
<td>59.0 – 60.0(59.5)</td>
<td>540</td>
</tr>
<tr>
<td>TOPAZ</td>
<td>52.0 – 55.0(53.3)</td>
<td>745</td>
</tr>
<tr>
<td>TASSO</td>
<td>38.4 – 46.8(43.5)</td>
<td>6434</td>
</tr>
<tr>
<td>TASSO</td>
<td>32.0 – 35.2(34.0)</td>
<td>52118</td>
</tr>
<tr>
<td>PLUTO</td>
<td>34.6(34.6)</td>
<td>6964</td>
</tr>
<tr>
<td>JADE</td>
<td>29.0 – 36.0(34.0)</td>
<td>12719</td>
</tr>
<tr>
<td>CELLO</td>
<td>34.0(34.0)</td>
<td>2600</td>
</tr>
<tr>
<td>MARKII</td>
<td>29.0(29.0)</td>
<td>5024</td>
</tr>
<tr>
<td>MARKII</td>
<td>29.0(29.0)</td>
<td>13829</td>
</tr>
<tr>
<td>MAC</td>
<td>29.0(29.0)</td>
<td>65000</td>
</tr>
<tr>
<td>TASSO</td>
<td>21.0 – 23.0(22.0)</td>
<td>1913</td>
</tr>
<tr>
<td>JADE</td>
<td>22.0(22.0)</td>
<td>1399</td>
</tr>
<tr>
<td>CELLO</td>
<td>22.0(22.0)</td>
<td>2000</td>
</tr>
<tr>
<td>TASSO</td>
<td>12.4 – 14.4(14.0)</td>
<td>2704</td>
</tr>
<tr>
<td>JADE</td>
<td>22.0(22.0)</td>
<td>2112</td>
</tr>
</tbody>
</table>
Perturbative expansion of EEC up to NNLO in terms of $\alpha_S$:

\[ \frac{1}{\sigma_t} \frac{d\Sigma(\chi, \mu)}{d \cos \chi} \bigg|_{\text{f.o.}} = \frac{\alpha_S(\mu)}{2\pi} \frac{d\tilde{A}(\chi, \mu)}{d \cos \chi} + \left( \frac{\alpha_S(\mu)}{2\pi} \right)^2 \frac{d\tilde{B}(\chi, \mu)}{d \cos \chi} + \left( \frac{\alpha_S(\mu)}{2\pi} \right)^3 \frac{d\tilde{C}(\chi, \mu)}{d \cos \chi} + \mathcal{O}(\alpha_S^4) \]

Comparison of EEC at various orders to OPAL data
The resummation of EEC @ NNLL was done by de Florian and Grazzini [de Florian et al. ’04]

\[
\left[ \frac{1}{\sigma_t} \frac{d\Sigma(\chi, \mu)}{d \cos \chi} \right]_{\text{res.}} = \frac{Q^2}{8} H(\alpha_S(\mu)) \int_0^\infty db \, b \, J_0(b \, Q \, \sqrt{y}) S(Q, b)
\]

with \( y = \sin^2 \frac{\chi}{2} \)

Large logs are exponentiated via the already seen Sudakov form factor:

\[
S(Q, b) = \exp \left\{ - \int_{b_0^2/b^2}^{Q^2} \frac{dq^2}{q^2} \left[ A(\alpha_S(q^2)) \log \frac{Q^2}{q^2} + B(\alpha_S(q^2)) \right] \right\}
\]

\[
A(\alpha_S) = \sum_{n=1}^{\infty} \left( \frac{\alpha_S}{4\pi} \right)^n A^{(n)}, \quad B(\alpha_S) = \sum_{n=1}^{\infty} \left( \frac{\alpha_S}{4\pi} \right)^n B^{(n)}, \quad H(\alpha_S) = 1 + \sum_{n=1}^{\infty} \left( \frac{\alpha_S}{4\pi} \right)^n H^{(n)}
\]

To retain NNLL accuracy we need: \( A^{(1)}, A^{(2)}, A^{(3)}, B^{(1)}, B^{(2)}, (H^{(1)}, H^{(2)}) \)
Resumed prediction describes the Sudakov peak well, starts to deviate from data at moderate angles $\implies$ have to match w/ fixed-order!
EEC with Resummation

Resumed prediction vs. data

Resumed prediction describes the Sudakov peak well, starts to deviate from data at moderate angles \( \Rightarrow \) have to match w/ fixed-order!

\[
\frac{1}{\sigma_t} \frac{d\Sigma}{d\chi} \text{ [1/rad]}
\]

\[
\chi \text{ [deg]}
\]

\[
Q = 91.2 \text{ GeV} \\
\alpha_s(Q) = 0.118
\]

- LL
- NLL
- NNLL
- OPAL
• In case of EEC only NNLL resummation is available
  \[\rightarrow\] R scheme cannot be used
• Modified R scheme can be used: determining the coefficient of the missing logs from the fixed-order calculation
• Fitting of the coefficients is challenging at NNLO
• When missing logs are present the log-R scheme is much more natural to use
• To be able to use the log-R scheme we have to define a cumulative observable
To define the cumulative distribution have to get rid of the divergence at large $\chi$:

$$\frac{1}{\sigma_t} \tilde{\Sigma}(\chi, \mu) \equiv \frac{1}{\sigma_t} \int_{0}^{\chi} d\chi' (1 + \cos \chi') \frac{d\Sigma(\chi', \mu)}{d\chi'} = \frac{1}{\sigma_t} \int_{0}^{y(\chi)} d\chi' 2(1 - y') \frac{d\Sigma(y', \mu)}{dy'}$$

In massless QCD:

$$\frac{1}{\sigma_t} \tilde{\Sigma}(\chi_{\text{max}}, \mu) = 1$$

From the cumulative distribution the original one can be easily obtained:

$$\frac{1}{\sigma_t} \frac{d\Sigma(\chi, \mu)}{d\chi} = \frac{1}{1 + \cos \chi} \frac{d}{d\chi} \left[ \frac{1}{\sigma_t} \tilde{\Sigma}(\chi, \mu) \right]$$
Matched EEC predictions using naive R and log-R matching with fixed-order NLO, bands: renorm. uncertainty $\mu_R \in [Q/2, 2Q]$
Matched EEC prediction using log-R matching with fixed-order NNLO, band: renorm. uncertainty $\mu_R \in [Q/2, 2Q]$
Matched EEC predictions using log-R scheme with N(N)LO, bands: renorm. uncertainty $\mu_R \in [Q/2, 2Q]$
Phenomenology
Considered \( Q = M_Z = 91.2 \text{ GeV} \)

Comparisons are made against OPAL and SLD data

Due to the low \( Q \) value, non-perturbative corrections are sizable:

1) Comparisons are made with N(N)LO + NNLL
2) N(N)LO + NNLL + analytic model for NP effects

Dokshitzer, et al., ‘99

\( \alpha_S(M_Z) \) value is extracted from various setups
### Fits without NP effects

<table>
<thead>
<tr>
<th>Fit range</th>
<th>NNLL+NLO ($R$) $\alpha_s(M_Z)$</th>
<th>$\chi^2$/d.o.f.</th>
<th>NNLL+NLO (log-$R$) $\alpha_s(M_Z)$</th>
<th>$\chi^2$/d.o.f.</th>
<th>NNLL+NNLO (log-$R$) $\alpha_s(M_Z)$</th>
<th>$\chi^2$/d.o.f.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0^\circ &lt; \chi &lt; 63^\circ$</td>
<td>0.133 ± 0.001</td>
<td>1.96</td>
<td>0.131 ± 0.003</td>
<td>1.21</td>
<td>0.129 ± 0.003</td>
<td>4.13</td>
</tr>
<tr>
<td>$15^\circ &lt; \chi &lt; 63^\circ$</td>
<td>0.132 ± 0.001</td>
<td>0.59</td>
<td>0.131 ± 0.003</td>
<td>0.54</td>
<td>0.128 ± 0.003</td>
<td>1.58</td>
</tr>
<tr>
<td>$15^\circ &lt; \chi &lt; 120^\circ$</td>
<td>0.135 ± 0.002</td>
<td>3.96</td>
<td>0.134 ± 0.004</td>
<td>5.12</td>
<td>0.127 ± 0.003</td>
<td>1.12</td>
</tr>
</tbody>
</table>

- Fit ranges were selected to comply with de Florian et al.
- Large hadronization corrections around Sudakov peak
  \[ \Rightarrow \text{large } \chi^2/\text{d.o.f.} \text{ & } \alpha_s \text{ value} \]
- Excluding the peak region the quality of fit only improves if NNLO is used in the matching
- R-matched NNLL+NLO results are in agreement with de Florian et al. slight change in value only due to difference in $A^{(3)}$ used
To incorporate non-perturbative effects the analytical model of Dokshitzer, Marchesini and Webber (’99) was used:

The Sudakov form factor is multiplied by:

$$S_{NP} = e^{-\frac{1}{2}a_1 b^2} (1 - 2a_2 b)$$

where $a_1$ and $a_2$ are free parameters of the NP model

$\Rightarrow$ the $\alpha_s$ fit becomes a three-parameter fit for $\alpha_s$, $a_1$, $a_2$

Fit range is selected to be between $0^\circ$ and $63^\circ$
NNLL+NLO+NP predictions for EEC fitted to OPAL and SLD data

Note the clear difference in shape between fit and data!
NNLL+NNLO+NP prediction for EEC fitted to OPAL and SLD data
Shape is described by prediction! One more `N' was needed in fixed-order!
\( \alpha_s \) values from fits using NP effects

**NNLL+NLO (R):**

\[
\alpha_s(M_Z) = 0.134^{+0.001}_{-0.009}, \quad a_1 = 1.55^{+4.26}_{-1.54} \text{ GeV}^2, \quad a_2 = -0.13^{+0.50}_{-0.05} \text{ GeV},
\]
\[
\chi^2 / \text{d.o.f.} = 0.81
\]

**NNLL+NLO (log-R):**

\[
\alpha_s(M_Z) = 0.128^{+0.002}_{-0.006}, \quad a_1 = 1.17^{+1.46}_{-0.29} \text{ GeV}^2, \quad a_2 = 0.13^{+0.14}_{-0.09} \text{ GeV},
\]
\[
\chi^2 / \text{d.o.f.} = 0.85
\]

**NNLL+NNLO:**

\[
\alpha_s(M_Z) = 0.121^{+0.001}_{-0.003}, \quad a_1 = 2.47^{+0.48}_{-2.38} \text{ GeV}^2, \quad a_2 = 0.31^{+0.27}_{-0.05} \text{ GeV}.
\]
\[
\chi^2 / \text{d.o.f.} = 1.18
\]
Outlook & Conclusions
Outlook

- The used NP model was very naive
- NP effects can be approximated with other methods:
  - NLO QCD calculations can be matched with parton shower
  - Through parton shower different hadronization models are usable:
    ▶ Cluster fragmentation model [Winter, Krauss and Soff, ’04]
    ▶ Lund string fragmentation model [Sjostrand, Mrenna and Skands, ’06]
- \( \alpha_s \) will be obtained using all data ever taken for EEC (see slide no. 32)
- In order to correctly assess uncertainties we teamed up with experimentalists
- The study is underway, to appear soon
  [AK, Kluth, Somogyi, Tulipant and Verbytskyi]
Conclusions

• NNLL+NNLO matching was performed for EEC
• log-R matching scheme was worked out for EEC
• $\alpha_S$ was obtained from fits to OPAL and SLD data both at NNLL+NLO and NNLL+NNLO accuracy at parton level and taking into account non-perturbative correction via an analytical model
• Found that adding an extra `N' in the fixed-order prediction is crucial to get good quality fits
• To be able to use naive R matching one more `N' is needed in the resummed formula
• Our best fit is: $\alpha_S(M_Z) = 0.121^{+0.001}_{-0.003}$, $\chi^2$/d.o.f. = 1.18
• A precise fit of $\alpha_S$ is on the way using modern tools to determine hadronization corrections
Thank you for your attention!