Hadronic total cross section, Wilson loop correlators and the QCD spectrum

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Rising Total Cross Sections

\[ \sigma_{tot}^{(pp)} = B \log^2 \frac{s}{s_M} + Z + Y_1 \left( \frac{s_M}{s} \right)^{\eta_1} - Y_2 \left( \frac{s_M}{s} \right)^{\eta_2} \]

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\[ \sigma^{(hh)}(s) \sim B \log^2 s \]

\( B \approx 0.3 \text{ mb universal} \), independent of the colliding hadrons

Consistent with Froissart bound (unitarity + mass gap) [Froissart (1961)]

\[ \sigma_{tot}^{(hh)}(s) \leq \frac{\pi}{m_T^2} \log^2 \frac{s}{s_0} \]
Total cross sections related to forward elastic amplitudes via optical theorem

\[ \sigma_{\text{tot}} \sim \frac{1}{s} \text{Im} \mathcal{M}(s, t = 0) \]

Soft high-energy hadron-hadron scattering: \( s \to \infty, |t| \leq 1 \text{GeV}^2 \)

One of the oldest unsolved problems of strong interactions

Impact parameter representation

\[ [t = -\vec{q}_\perp^2] \]

\[ \mathcal{M}(s, t) = i \, 2s \int d^2 \vec{b}_\perp e^{i\vec{q}_\perp \cdot \vec{b}_\perp} A(s, \vec{b}_\perp) \]

\[ \sigma_{\text{tot}} = 2 \Re \int d^2 \vec{b}_\perp A(s, \vec{b}_\perp) \]

Unitarity: \( |A(s, \vec{b}_\perp) - 1| \leq 1 \)
Total cross sections related to forward elastic amplitudes via optical theorem

\[ \sigma_{\text{tot}} \underset{s \to \infty}{\approx} \frac{1}{s} \Im \mathcal{M}(s, t = 0) \]

Soft high-energy hadron-hadron scattering: \( s \to \infty, \ |t| \leq 1 \text{GeV}^2 \)
One of the oldest unsolved problems of strong interactions

Impact parameter representation for unpolarised scattering \([t = -\vec{q}^2] \]

\[ \mathcal{M}(s, t) = i \ 4\pi s \int_0^\infty dbb \ J_0(bq) A(s, b) \]

\[ \sigma_{\text{tot}} = 4\pi \Re \int_0^\infty dbb \ A(s, b) \]

Unitarity: \[ |A(s, b) - 1| \leq 1 \]
How to Obtain a Rising Total Cross Section

Typical shape: \( A \to 0 \) at large \( b \), for \( b > b_c(s) \) the amplitude is “negligible”

\[
\sigma_{\text{tot}} \sim b_c(s)^2: \text{ how does } b_c \text{ change with } s? 
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How to Obtain a Rising Total Cross Section

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$\sigma_{tot} \rightarrow 0$
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\[
\sigma_{\text{tot}} \sim b_c(s)^2: \text{ how does } b_c \text{ change with } s? \quad b_c(s) \to \text{const.}
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$\sigma_{\text{tot}} \sim b_c(s)^2$: how does $b_c$ change with $s$? $b_c(s) \to \infty$

$$\sigma_{\text{tot}} = 4\pi b_c(s)^2 \Re \int_0^\infty dx \times A(s, b_c(s)x) \to 4\pi b_c(s)^2 C$$
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\[ \sigma_{\text{tot}} = 4\pi b_c(s)^2 \Re \int_0^\infty dx \times A(s, b_c(s) x) \to 4\pi b_c(s)^2 C \]

- $b_c(s)$ gives the energy dependence
- large $b$ ($\gg m^{-1}$) is relevant
QCD: fundamental theory, should explain the rise of total cross sections

$$|t| \lesssim 1 \text{GeV}^2$$, PT not fully reliable \(\rightarrow\) NP approach \([Nachtmann\ (1991)]\)

1. **Partonic description of hadrons over a small time–window** (~ 2 fm)
2. **Partons do not split or annihilate**, treated as in/out states of a scattering process
3. **Lightlike trajectories approx. unchanged** in the process, only soft gluon exchange
4. **Hadronic amplitude after folding with hadronic wave function**
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Nonperturbative Approach

Partonic scattering amplitudes from the correlation function of infinite lightlike Wilson lines [Nachtmann (1991)]

To avoid IR divergences $\rightarrow$ hadronic amplitudes

- mesons as wave packets of transverse colourless dipoles
- dipole scattering amplitudes from the correlation function of infinite lightlike Wilson loops [Dosch et al. (1996)]

Intermediate regularisation: finite hyperbolic angle $\chi$ and length $2T$ [Verlinde, Verlinde (1993)]

Extends to baryon-baryon scattering adopting a quark-diquark description [Rueter, Dosch (1996)]
Elastic meson-meson from dipole-dipole scattering \cite{Dosch et al. (1996)}

\[ A(s, \vec{b}_\perp) = \langle\langle A^{(dd)}(s, \vec{b}_\perp; \nu_1, \nu_2)\rangle\rangle \]

\( \nu_i = (f_i, \vec{R}_{i\perp}) \), \( f_i \) longitudinal momentum fraction, \( \vec{R}_{i\perp} \) transverse size

\( \langle\langle . . . \rangle\rangle \): average over \( \nu_{1,2} \) with mesonic wave functions, \( \langle\langle 1 \rangle\rangle = 1 \)

\textit{dd} scattering amplitude in \( b \)-space \( \leftrightarrow \) Wilson-loop correlation function

\[ -A^{(dd)}(s, \vec{b}_\perp; \nu_1, \nu_2) = \lim_{\chi \to \infty} C_M(\chi; \vec{b}_\perp, \nu_1, \nu_2) \]

\[ \chi \overset{\sim}{=} \log \frac{s}{m^2} \]

\[ G_M(\chi; T; \vec{b}_\perp, \nu_1, \nu_2) \equiv \frac{\langle \mathcal{W}_{C_1} \mathcal{W}_{C_2} \rangle}{\langle \mathcal{W}_{C_1} \rangle \langle \mathcal{W}_{C_2} \rangle} - 1 \]

\[ C_M \equiv \lim_{T \to \infty} G_M \]

\( \langle . . . \rangle \): expectation value in the functional integral formalism
Wilson Loop Correlation Function

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Wilson Loop Correlation Function

NP techniques available in Euclidean space $\Rightarrow$ Euclidean formulation

$[\text{Meggiolaro (1997), Meggiolaro (2005)}]$

$$\mathcal{G}_E(\theta; T; \vec{b}_\perp, \nu_1, \nu_2) \equiv \frac{\langle \mathcal{W}_{C_1} \mathcal{W}_{C_2} \rangle}{\langle \mathcal{W}_{C_1} \rangle \langle \mathcal{W}_{C_2} \rangle} - 1,$$

$$C_E \equiv \lim_{T \to \infty} \mathcal{G}_E$$
Analytic Continuation to Euclidean Space

Analytic continuation relations [Meggiolaro (2005), MG, Meggiolaro (2009)]

\[ C_M(\chi) = C_E(\theta \rightarrow -i\chi) \]

AC + Euclidean symmetries \(\Rightarrow\) crossing relations [MG, Meggiolaro (2006)]

\[ C_M(i\pi - \chi; \vec{R}_1\perp, \vec{R}_2\perp) = C_M(\chi; \vec{R}_1\perp, -\vec{R}_2\perp) \]
Nonperturbative Models

Euclidean formulation opens the way to NP techniques:

- **Stochastic Vacuum Model** [Berger, Nachtmann (1999), Shoshi et al. (2003)]
- **Instanton Liquid Model** [Shuryak, Zahed (2000), MG, Meggiolaro (2010)]
- **AdS/CFT Correspondence** [Janik, Peschanski (2000a,b), MG, Peschanski (2010)]
- **Lattice Gauge Theory** [MG, Meggiolaro (2008), MG, Meggiolaro (2010)]

### Formulas

**Stochastic Vacuum Model (SVM)**

\[ C_E = \frac{2}{3} e^{-\frac{1}{3} \cot \theta K_{SVM}} + \frac{1}{3} e^{\frac{2}{3} \cot \theta K_{SVM}} - 1 \]

**Instanton Liquid Model (ILM)**

\[ C_E = \frac{K_{ILM}}{\sin \theta} \]

**Perturbation Theory (PT)**

\[ C_E = K_{PT} \cot^2 \theta \]

**ILM + PT (ILMp)**

\[ C_E = \frac{K_{ILMp}}{\sin \theta} + K'_{ILMp}(\cot \theta)^2 \]

**AdS/CFT correspondence**

\[ C_E = e^{\frac{K_{AdS}}{\sin \theta}} + K'_{AdS} \cot \theta + K''_{AdS} \cos \theta \cot \theta - 1 \]
Lattice calculations give “true” prediction of QCD (within errors) ⇒ test analytic NP calculations

Are the analytic NP calculations compatible with the lattice results?

- SVM/ILM do not match/fit well the data and $\sigma_{\text{SVM,ILM}}^{\text{tot}} \rightarrow \text{const.}$ as $s \rightarrow \infty$.
- ILM+PT gives improved best fits but $\sigma_{\text{ILM+PT}}^{\text{tot}} \rightarrow \text{const.}$ as $s \rightarrow \infty$.
- AdS/CFT: $\sigma_{\text{tot}} \propto s^{\frac{1}{3}}$ but for onium-onium scattering in $\mathcal{N} = 4$ SYM [MG, Peschanski (2010)]

Are the lattice results compatible with rising total cross sections?

- More general fits, but care is needed because of the AC
- Constrain admissible fitting functions with physical requirements (unitarity, crossing symmetry, . . . )
- Parameterisations fitting well the data and leading to rising total cross sections exist [MG, Meggiolaro, Moretti (2012)]
NP Models, Lattice Results and Rising Cross Sections

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Are the analytic NP calculations compatible with the lattice results?

- SVM/ILM do not match/fit well the data and \( \sigma_{tot}^{SVM,ILM} \rightarrow const. \) as \( s \rightarrow \infty \).
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Summary and Questions

- $\sigma_{\text{tot}} \sim$ large-$b$ behaviour of elastic scattering amplitudes in impact-parameter space $A(s, b)$
- QCD at large-$s$ and small-$t$: $A(s, b) \sim$ Wilson loop correlation function
- Analytic models fail to reproduce the lattice data and to capture the rising behaviour of $\sigma_{\text{tot}}$
- Lattice data compatible with rising behaviour, but large arbitrariness in the parameterisations

1. What are the large-$s$ and large-$b$ behaviour of $A(s, b)$?
2. What sets the physical scale in $\sigma_{\text{tot}}$?
3. How does $\sigma_{\text{tot}}$ relate to the hadronic spectrum?
Relating Total Cross Sections and the QCD Spectrum

How to extract $\theta$ and $b$ dependencies?

Basic idea: insert a complete set of states between the Wilson loops

$$\langle 0 | O_1(t) O_2(0) | 0 \rangle = \sum_n e^{-E_n t} \langle 0 | O_1(0) | n \rangle \langle n | O_2(0) | 0 \rangle$$

Complications: nonlocal operators, nontrivial angular dependence

Use asymptotic states with simple transformation properties

$$|\alpha\rangle = |\{n_a(\alpha)\}, \{\vec{p}\}, \{s_3\}\rangle$$

$\{n_a(\alpha)\}$: particle content, $\{\vec{p}\}$: momenta, $\{s_3\}$: 3rd component of spin

$$\sum_n |n\rangle \langle n| = \sum_\alpha \mathcal{P}_\alpha \sum_{\{s_3\}_\alpha} \int d\Omega_\alpha |\alpha\rangle \langle \alpha|$$

$$\mathcal{P}_\alpha = \frac{1}{\prod_a n_a(\alpha)!} \text{ symmetry factor, } d\Omega_\alpha \text{ phase space measure}$$
Sketch of Derivation 1

Rotate Euclidean time along the impact parameter (equivalent description)

\[ G_E(\theta; T; \vec{b}_\perp, \nu_1, \nu_2) \equiv \frac{\langle \mathcal{W}_C^1 \mathcal{W}_C^2 \rangle}{\langle \mathcal{W}_C^1 \rangle \langle \mathcal{W}_C^2 \rangle} - 1 \]
Sketch of Derivation 1

Rotate Euclidean time along the impact parameter (equivalent description)

\[ \tilde{G}_E(\theta; T; b; \nu_1, \nu_2) \equiv \frac{\langle \mathcal{W}_C \mathcal{W}_{\tilde{C}} \rangle}{\langle \mathcal{W}_{\tilde{C}} \rangle \langle \mathcal{W}_C \rangle} - 1 \]
Use Wilson loop operators

\[ \tilde{G}_E(\theta; T; b; \nu_1, \nu_2) \equiv \frac{\langle \mathcal{W}\tilde{C}_1 \mathcal{W}\tilde{C}_2 \rangle}{\langle \mathcal{W}\tilde{C}_1 \rangle \langle \mathcal{W}\tilde{C}_2 \rangle} - 1 = \frac{\langle 0 | T \{ \hat{\mathcal{W}}\tilde{C}_1, \hat{\mathcal{W}}\tilde{C}_2 \} | 0 \rangle}{\langle 0 | \hat{\mathcal{W}}\tilde{C}_1 | 0 \rangle \langle 0 | \hat{\mathcal{W}}\tilde{C}_2 | 0 \rangle} - 1 \]
Consider loops with no temporal overlap ($b > b_0$)

$$\tilde{G}_E(\theta; T; b; \nu_1, \nu_2) \equiv \frac{\langle \mathcal{W}\tilde{C}_1 \mathcal{W}\tilde{C}_2 \rangle}{\langle \mathcal{W}\tilde{C}_1 \rangle \langle \mathcal{W}\tilde{C}_2 \rangle} - 1 = \frac{\langle 0|\hat{\mathcal{W}}\tilde{C}_1 \hat{\mathcal{W}}\tilde{C}_2 |0 \rangle}{\langle 0|\hat{\mathcal{W}}\tilde{C}_1 |0 \rangle \langle 0|\hat{\mathcal{W}}\tilde{C}_2 |0 \rangle} - 1$$
Sketch of Derivation 4

Insert a complete set of states

\[ \tilde{G}_E(\theta; T; b; \nu_1, \nu_2) = \frac{\langle 0|\hat{W}_{\tilde{C}_1} \hat{W}_{\tilde{C}_2}|0\rangle}{\langle 0|\hat{W}_{\tilde{C}_1}|0\rangle \langle 0|\hat{W}_{\tilde{C}_2}|0\rangle} - 1 = \sum_{n \neq 0} \frac{\langle 0|\hat{W}_{\tilde{C}_1}|n\rangle \langle n|\hat{W}_{\tilde{C}_2}|0\rangle}{\langle 0|\hat{W}_{\tilde{C}_1}|0\rangle \langle 0|\hat{W}_{\tilde{C}_2}|0\rangle} \]
Sketch of Derivation 5

Rotate around 3-axis and translate centres to the origin, $|n_{\pm \frac{\theta}{2}}\rangle = e^{\pm i \hat{J}_3 \frac{\theta}{2}} |n\rangle$

$$\tilde{G}_E(\theta; T; b; \nu_1, \nu_2) = \sum_{n \neq 0} e^{-bE_n} e^{i \theta S_3 n} \frac{\langle 0 | \hat{\mathcal{W}}_0(\nu_1) | n_{\frac{\theta}{2}} \rangle \langle n_{-\frac{\theta}{2}} | \hat{\mathcal{W}}_0(\nu_2) | 0 \rangle}{\langle 0 | \hat{\mathcal{W}}_0(\nu_1) | 0 \rangle \langle 0 | \hat{\mathcal{W}}_0(\nu_2) | 0 \rangle}$$
Sketch of Derivation 5

Rotate around 3-axis and translate centres to the origin, \( |n_{\pm\frac{\theta}{2}}\rangle = e^{\pm i\hat{J}_3\frac{\theta}{2}} |n\rangle \)

\[
\tilde{G}_E(\theta; T; b; \nu_1, \nu_2) = \sum_{n \neq 0} e^{-bE_n} e^{i\theta S_3 n} \frac{\langle 0|\hat{W}_0(\nu_1) |n_{\theta/2}\rangle \langle n_{-\theta/2}|\hat{W}_0(\nu_2)|0\rangle}{\langle 0|\hat{W}_0(\nu_1)|0\rangle \langle 0|\hat{W}_0(\nu_2)|0\rangle}
\]
Take $T \to \infty$

$$\tilde{C}_E(\theta; b; \nu_1, \nu_2) = \sum_{n \neq 0} e^{-bE_n} e^{i\theta S_{3n}} \frac{\langle 0|\hat{W}_0(\nu_1)|n_{\theta} \rangle \langle n_{-\theta}^2|\hat{W}_0(\nu_2)|0\rangle}{\langle 0|\hat{W}_0(\nu_1)|0\rangle \langle 0|\hat{W}_0(\nu_2)|0\rangle}$$

$$= \sum_{\alpha \neq 0} P_\alpha \sum_{\{s_3\}_\alpha} e^{i\theta S_{3\alpha}} \int d\Omega_\alpha e^{-bE_\alpha} W_\alpha^+ (\{\vec{p}_{\theta/2}\}, \{s_3\}; \nu_1) W_\alpha^- (\{\vec{p}_{-\theta/2}\}, \{s_3\}; \nu_2)$$

$E_\alpha, S_{3\alpha}$: total energy and 3rd component of spin in state $\alpha$

$\{\vec{p}_{\pm \theta/2}\}$: all momenta rotated around 3-axis

Selection rule: $W_\alpha^\pm$ nonzero only for vanishing discrete charges (electric charge, baryon number, strangeness, . . . )

$$Q = B = S = \ldots = 0$$
Sketch of Derivation 7

Change of variables (\sim momentum components along the original loops)

\[ x_\pm = (\vec{p}_{\pm \theta})_1 = \cos \frac{\theta}{2} p_1 \pm \sin \frac{\theta}{2} p_2 \quad \vec{p}_{\pm \theta} = (x_\pm, \pm \cot \theta x_\pm \mp \frac{1}{\sin \theta} x_\mp, p_3) \]

\[ \tilde{C}_E(\theta; b; \nu_1, \nu_2) = \sum_{\alpha \neq 0} \mathcal{P}_\alpha \sum_{\{s_3\}_\alpha} e^{i\theta S_{3\alpha}} \int d\Omega'_\alpha e^{-bE_\alpha} \]

\[ \times \mathcal{W}^+ \left( \{(x_+, \cot \theta x_+ - \frac{1}{\sin \theta} x_-, p_3)\}, \{s_3\}; \nu_1 \right) \]

\[ \times \mathcal{W}^- \left( \{(x_-, -\cot \theta x_- + \frac{1}{\sin \theta} x_+, p_3)\}, \{s_3\}; \nu_2 \right) \]

\[ d\Omega_\alpha = \prod_{a,i} \left[ \frac{d^3p}{(2\pi)^3 2\varepsilon} \right]_{a,i} \rightarrow \prod_{a,i} \frac{1}{\sin \theta} \left[ \frac{dx_+ dx_- dp_3}{(2\pi)^3 2\varepsilon} \right]_{a,i} = \frac{d\Omega'_\alpha}{(\sin \theta)^{\mathcal{N}_\alpha}} \]

\[ \varepsilon = \sqrt{m^2 + \vec{p}^2} = \sqrt{m^2 + \left( \frac{x_+ + x_-}{2 \cos(\theta/2)} \right)^2 + \left( \frac{x_+ - x_-}{2 \sin(\theta/2)} \right)^2 + p_3^2} = \varepsilon(\theta) \]
Assumptions:

1. The analytic continuation can be performed term by term
   [crucial assumption, requires good convergence properties]

2. $W^{\pm}_{\alpha}$ expressed in terms of $x_{\pm}$ are analytic in $\theta$
   [$\delta(\sum x_{+})$ and $\delta(\sum x_{-})$ in $W^{\pm}_{\alpha}$ due to translation invariance]

Complex $z = \theta - i\chi$

\[
\frac{e^{izS_{3\alpha}}}{(\sin z)^{N_{\alpha}}} \int d\Omega' e^{-bE_{\alpha}} W^{+}_{\alpha} \left( \{(x_{+}, \cot z x_{+} - \frac{1}{\sin z} x_{-}, p_{3})\}, \{s_{3}\}; \nu_{1} \right) \\
\times W^{-}_{\alpha} \left( \{(x_{-}, -\cot z x_{-} + \frac{1}{\sin z} x_{+}, p_{3})\}, \{s_{3}\}; \nu_{2} \right)
\]
Analytic Continuation and Large-$\chi$ Limit

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Complex $z = \theta - i\chi$

$$e^{izS_{3\alpha}} \int \frac{d\Omega_{\alpha}^{\prime} e^{-bE_{\alpha}} W_{\alpha}^{+}(\{ (x_{+}, \cot z x_{+} - \frac{1}{\sin z} x_{-}, p_{3} ) \}, \{ s_{3} \}; \nu_{1})}{(\sin z)^{N_{\alpha}}} \times W_{\alpha}^{-}(\{ (x_{-}, - \cot z x_{-} + \frac{1}{\sin z} x_{+}, p_{3} ) \}, \{ s_{3} \}; \nu_{2})$$

Integration is convergent as $\Re \varepsilon > 0$ for $\theta \in (0, \pi)$ [$E_{\alpha} = \sum \varepsilon$]
Analytic Continuation and Large-χ Limit

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   \[ \delta(\sum x_{+}) \text{ and } \delta(\sum x_{-}) \text{ in } W_{\alpha}^{\pm} \text{ due to translation invariance} \]

Complex \( z = \theta - i\chi \)

\[
e^{izS_{3\alpha}} \frac{1}{(\sin z)^{N_{\alpha}}} \int d\Omega^{\prime}_{\alpha} e^{-bE_{\alpha}} W_{\alpha}^{+}(\{(x_{+}, \cot zx_{+} - \frac{1}{\sin z} x_{-}, p_{3})\}, \{s_{3}\}; \nu_{1})
\]
\[
	imes W_{\alpha}^{-}(\{(x_{-}, -\cot zx_{-} + \frac{1}{\sin z} x_{+}, p_{3})\}, \{s_{3}\}; \nu_{2})
\]

Take \( \theta \to 0 \)
Analytic Continuation and Large-$\chi$ Limit

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   [$\delta(\sum x_{+})$ and $\delta(\sum x_{-})$ in $W_{\pm}^{\alpha}$ due to translation invariance]

Complex $z = \theta - i\chi$

$$i^{N_{\alpha}}e^{\chi S_{3\alpha}}/(\sinh \chi)^{N_{\alpha}} \int d\Omega' e^{-bE_{\alpha}} W_{\alpha}^{+}(\{(x_{+}, i\coth \chi x_{+} - i\frac{1}{\sinh \chi} x_{-}, p_{3})\}, \{s_{3}\}; \nu_{1})$$

$$\times W_{\alpha}^{-}(\{(x_{-}, -i\coth \chi x_{-} + i\frac{1}{\sinh \chi} x_{+}, p_{3})\}, \{s_{3}\}; \nu_{2})$$

Take $\theta \to 0$, $\chi \to \infty$
Analytic Continuation and Large-$\chi$ Limit

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1. the analytic continuation can be performed term by term
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2. $W_{\alpha}^{\pm}$ expressed in terms of $x_{\pm}$ are analytic in $\theta$
   [$\delta(\sum x_{+})$ and $\delta(\sum x_{-})$ in $W_{\alpha}^{\pm}$ due to translation invariance]

Complex $z = \theta - i\chi$

$$(2i)^{N_{\alpha}} e^{\chi(S_{3\alpha} - N_{\alpha})} \int d\Omega'_{\alpha} e^{-\tilde{E}_{\alpha}} W_{\alpha}^{+}(\{(x_{+}, ix_{+}, p_{3})\}, \{s_{3}\}; \nu_{1})$$

$$\times W_{\alpha}^{-}(\{(x_{-}, -ix_{-}, p_{3})\}, \{s_{3}\}; \nu_{2})$$

Take $\theta \rightarrow 0$, $\chi \rightarrow \infty$: integrand independent of $\chi$

For $\chi \rightarrow \infty$, $\varepsilon \rightarrow \sqrt{m^2 + p_{3}^2} = \tilde{\varepsilon}$, independent of $\chi$; set $\tilde{E}_{\alpha} = \sum \tilde{\varepsilon}$
Analytic Continuation and Large-$\chi$ Limit

Assumptions:

1. The analytic continuation can be performed term by term [crucial assumption, requires good convergence properties]

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Complex $z = \theta - i\chi$

$$\begin{align*}
(2i)^{N_{\alpha}} e^{\chi(S_{3\alpha} - N_{\alpha})} & \int d\tilde{\Omega}_{\alpha} e^{-b\tilde{E}_{\alpha}} \int dX_{\alpha}^{+} W_{\alpha}^{+}(\{(x_{+}, ix_{+}, p_{3})\}, \{s_{3}\}; \nu_{1}) \\
& \times \int dX_{\alpha}^{-} W_{\alpha}^{-}(\{(x_{-}, -ix_{-}, p_{3})\}, \{s_{3}\}; \nu_{2}) \\
\end{align*}$$

$$d\tilde{\Omega}_{\alpha} = \prod_{a,i} \left[ \frac{dp_{3}}{(2\pi)^{2\varepsilon}} \right]_{a,i} \quad dX_{\alpha}^{\pm} = \prod_{a,i} \left[ \frac{dx_{\pm}}{(2\pi)^{2\varepsilon}} \right]_{a,i}$$
Analytic Continuation and Large-\(\chi\) Limit

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2. \(W^\pm_\alpha\) expressed in terms of \(x^\pm\) are analytic in \(\theta\) 
   \([\delta(\sum x^+) \text{ and } \delta(\sum x^-) \text{ in } W^\pm_\alpha \text{ due to translation invariance}]

Complex \(z = \theta - i\chi\)

\[
(2i)^{N_\alpha} e^{\chi(S^\text{max}_{3\alpha} - N_\alpha)} \int d\tilde{\Omega}_\alpha e^{-b\tilde{E}_\alpha} \int dX^+_\alpha W^+_\alpha (\{(x^+, ix^+, p_3)\}, \{s\}; \nu_1) \times \int dX^-_\alpha W^-_\alpha (\{(x^-, -ix^-, p_3)\}, \{s\}; \nu_2)
\]

Leading contribution at large \(\chi\) from maximal \(S^\text{max}_{3\alpha} = \sum_a n_a(\alpha)s^{(a)}\), with \(s^{(a)i} = s^{(a)}\) for all particles
Large-$b$ Expansion

We are interested in $b \gg m^{-1}$: set $p_3 = \frac{\tilde{p}_3}{\sqrt{bm}}$

$$\tilde{\varepsilon} = \sqrt{m^2 + p_3^2} = m \left( 1 + \frac{1}{2bm} \left( \frac{\tilde{p}_3}{m} \right)^2 + \mathcal{O} \left( \frac{1}{(bm)^2} \right) \right)$$

$$\frac{dp_3}{2\pi \tilde{\varepsilon}} = \frac{d\tilde{p}_3}{4\pi m \sqrt{bm}} \left( 1 + \mathcal{O} \left( \frac{1}{bm} \right) \right)$$

$$\int dX^\pm_\alpha W^\pm_\alpha \left( \{ x_\pm, \pm ix_\pm, \frac{\tilde{p}_3}{\sqrt{bm}} \} \right), \{ s_3^{(a)} \}; \nu) = \mathcal{F}^\pm_\alpha (\nu) + \ldots$$

For large $b$ the integration in $d\tilde{p}_3$ becomes Gaussian

$$\frac{dp_3}{2\pi \tilde{\varepsilon}} e^{-b\tilde{\varepsilon}} \rightarrow \frac{d\tilde{p}_3}{4\pi m \sqrt{bm}} e^{-bm - \frac{1}{2} \left( \frac{\tilde{p}_3}{m} \right)^2} = \frac{e^{-bm}}{2\sqrt{2\pi bm}}$$

If $\mathcal{F}^\pm_\alpha (\nu_{1,2})$ vanish $\rightarrow$ extra inverse powers of $bm$ and constant factors
Asymptotic Behaviour of the Correlator

At large $\chi$ and large $b$, setting $\tilde{C}_M(\chi; b; \nu_1, \nu_2) \equiv \tilde{C}_E(-i\chi; b; \nu_1, \nu_2)$

$$\tilde{C}_M(\chi; b; \nu_1, \nu_2) \sim \sum_{\chi \to \infty, b \to \infty} \mathcal{P}_\alpha i^{N_\alpha} \mathcal{F}_\alpha^+(\nu_1) \mathcal{F}_\alpha^-(\nu_2) \prod_a w_a n_a(\alpha)$$

up to $\mathcal{O}(e^{-\chi})$ and $\mathcal{O}(b^{-1})$

$$w_a(\chi, b) = \frac{1}{\sqrt{2\pi bm(a)}} e^{\chi [s^{(a)} - 1]} e^{-bm(a)} = \frac{1}{\sqrt{2\pi bm(a)}} e^{[R_{\text{eff}}^{(a)}(s) - b]m(a)}$$

Reminiscent of exchange of spin-$J$ particle $\rightarrow$ contribution $\propto s^{J-1}$

Contribution of state $\alpha$ non-negligible only for

$$b \lesssim R_{\text{eff}}^{[\alpha]} = \frac{\sum_a n_a(\alpha) m^{(a)} R^{(a)}(s)}{\sum_a n_a(\alpha) m^{(a)}} \quad R^{(a)}(s) \equiv \frac{s^{(a)} - 1}{m^{(a)}}$$
\( \tilde{C}_M \) enters the expression for \( \sigma_{\text{tot}} \):
\[
\sigma_{\text{tot}} = 2 \text{Re} \int d^2 \vec{b}_\perp A(s, \vec{b}_\perp) = -4\pi \text{Re} \int_0^\infty dbb \langle\langle \tilde{C}_M(\chi; b; \nu_1, \nu_2) \rangle\rangle
\]

What is the characteristic \( b_c \)?
\[
b_c(s) = \max_{\alpha} R_{\text{eff}}^{[\alpha]}(s) = \max_a R_{\text{eff}}^{(a)}(s) = \left[ \max_a \frac{s^{(a)} - 1}{m^{(a)}} \right] \chi = \frac{\tilde{s} - 1}{\tilde{m}} \chi
\]

If higher-spin \( (s^{(a)} > 1) \) stable states exist
\[
\sigma_{\text{tot}} \propto b_c(s)^2 \sim \left( \frac{\tilde{s} - 1}{\tilde{m}} \right)^2 \log^2 s
\]

If \( \tilde{s} < 1 \), \( \sigma_{\text{tot}} \to 0; \) if \( \tilde{s} = 1 \), \( \sigma_{\text{tot}} \to \text{const.} \)
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\]

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\sigma_{\text{tot}} \propto b_c(s)^2 \sim \left( \frac{\tilde{s} - 1}{\tilde{m}} \right)^2 \log^2 s
\]

If \( \tilde{s} < 1 \), \( \sigma_{\text{tot}} \to 0 \); if \( \tilde{s} = 1 \), \( \sigma_{\text{tot}} \to \text{const} \).
Scaling Variable

Can we get something more? Change variables to

\[ z = \frac{1}{\sqrt{\chi}} e^{\chi(\tilde{s} - 1)} e^{-\tilde{m}b} \]

\[ \frac{dz}{z} = -\tilde{m}db \]

and take \( \chi \to \infty \) with \( z \) fixed

\[ w_a(\chi, z) = \exp \left\{ m^{(a)} \chi \left[ \frac{s^{(a)} - 1}{m^{(a)}} - \frac{\tilde{s} - 1}{\tilde{m}} \right] \right\} \]

\[ \sqrt{2\pi} \frac{m^{(a)}}{\tilde{m}} \chi^{-\frac{m^{(a)}}{\tilde{m}}} \log \frac{e^{\chi(\tilde{s} - 1)}}{\sqrt{\chi z}} \]

\[ \rightarrow \begin{cases} 0 & (m^{(a)}, s^{(a)}) \neq (\tilde{m}, \tilde{s}) \\ \frac{z}{\sqrt{2\pi[\tilde{s} - 1]}} & (m^{(a)}, s^{(a)}) = (\tilde{m}, \tilde{s}) \end{cases} \]

Only states containing particle/antiparticle \( (\tilde{m}, \tilde{s}) \) survive
Large-$\chi, b$ behaviour encoded in

\[ \lim_{\chi, b \to \infty} \tilde{C}_M(\chi; b; \nu_1, \nu_2) \bigg|_{z \text{ fixed}} = g(z; \nu_1, \nu_2) - 1 \]

\( \langle \tilde{C}_M \rangle \) bounded (unitarity) for all physical processes \( \to \) expect \( \tilde{C}_M \) bounded

\( \to |g(z; \nu_1, \nu_2)| \leq 1 \) bounded \( \forall z, \nu_1, \nu_2 \)

\[ \sigma_{tot} \approx \frac{4\pi}{\bar{m}^2} \text{Re} \langle J \rangle \quad J = \int_0^{e^{-\eta}} \frac{dz}{z} \log \frac{e^{-\eta}}{z} [1 - g(z; \nu_1, \nu_2)] \]

\[ \eta = \chi(\tilde{s} - 1) - \frac{1}{2} \log \chi \sim (\tilde{s} - 1) \log s \]

Look for \( O(\eta^2) \) terms in \( J \)

\[ J = \frac{1}{2} \eta^2 [1 - g_\infty(\nu_1, \nu_2)] + O(\eta) \]
Universal “Froissart-like” Total Cross Section

\[ \sigma_{\text{tot}} \sim \left. \frac{2\pi(\tilde{s} - 1)^2}{\tilde{m}^2} \right|_{s \to \infty} \left[ 1 - \text{Re} \langle \langle g_{\infty}(\nu_1, \nu_2) \rangle \rangle \right] \eta^2 + \mathcal{O}(\eta) \]

Bound on \( \sigma_{\text{tot}} \)

\[ \sigma_{\text{tot}} \leq \left. \frac{4\pi(\tilde{s} - 1)^2}{\tilde{m}^2} \log^2 \frac{s}{m^2} \right|_{s \to \infty} \]

If \( \text{Re} \, g_{\infty}(\nu_1, \nu_2) = 0 \) (“black disk assumption”), \( \sigma_{\text{tot}} \) is universal and the leading \( \mathcal{O}(\eta^2) \) term is entirely determined by the spectrum

\[ \sigma_{\text{tot}} \sim \left. \frac{2\pi(\tilde{s} - 1)^2}{\tilde{m}^2} \log^2 \frac{s}{m^2} \right|_{s \to \infty} \]

True also if \( g \) oscillates at infinity, unaffected by small-\( b \) behaviour.
Universal “Froissart-like” Total Cross Section

\[ \sigma_{\text{tot}} \underset{s \to \infty}{\sim} \frac{2\pi(\tilde{s} - 1)^2}{\tilde{m}^2} [1 - \text{Re} \langle \langle g_\infty(\nu_1, \nu_2) \rangle \rangle] \eta^2 + O(\eta) \]

Bound on \( \sigma_{\text{tot}} \)

\[ \sigma_{\text{tot}} \underset{s \to \infty}{\ll} \frac{4\pi(\tilde{s} - 1)^2}{\tilde{m}^2} \log^2 \frac{s}{m^2} \]

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Universal “Froissart-like” Total Cross Section

\[ \sigma_{\text{tot}} \sim \frac{2\pi (\tilde{s} - 1)^2}{\tilde{m}^2} [1 - \text{Re} \langle \langle g_\infty (\nu_1, \nu_2) \rangle \rangle] \eta^2 + O(\eta) \]

Bound on \( \sigma_{\text{tot}} \)

\[ \sigma_{\text{tot}} \lesssim \frac{4\pi (\tilde{s} - 1)^2}{\tilde{m}^2} \log^2 \frac{s}{m^2} \]

If \( \text{Re} g_\infty (\nu_1, \nu_2) = 0 \) (“black disk assumption”), \( \sigma_{\text{tot}} \) is universal and the leading \( O(\eta^2) \) term is entirely determined by the spectrum

\[ \sigma_{\text{tot}} \sim \frac{2\pi (\tilde{s} - 1)^2}{\tilde{m}^2} \log^2 \frac{s}{m^2} \]

True also if \( g \) oscillates at infinity, unaffected by small-\( b \) behaviour
Maximise $\frac{s^{(a)} - 1}{m^{(a)}}$ over asymptotic stable states of QCD in isolation

Data from [Nubase (2003), Gregory at al. (2012)]

$\Omega^\pm$ baryon, $m_{\Omega^\pm} \simeq 1.67$ GeV, $J^P = \frac{3}{2}^+$, $|Q| = 1$, $|S| = 3$

$B_{\text{exp}} \simeq 0.69 \div 0.73$ GeV$^{-2}$ vs. $B_{\text{th}} \simeq 0.56$ GeV$^{-2}$

Glueball spectrum: $B_Q \gtrsim 1.6B_{\text{exp}}$, large unquenching effects?
“Froissart-like” Bound

Froissart-Łukaszuk-Martin bound

\[
\lim_{s \to \infty} \frac{\sigma_{\text{tot}}}{\log^2 \frac{s}{m^2}} \leq \frac{\pi}{m^2_{\pi}} \simeq 59 \text{ mb}
\]

Our “Froissart-like” bound is much more restrictive

\[
\lim_{s \to \infty} \frac{\sigma_{\text{tot}}}{\log^2 \frac{s}{m^2}} \leq 2B_{\text{th}} = \frac{\pi}{m^2_{\Omega}} \simeq 0.44 \text{ mb}
\]

In the \(N_f = 2\) chiral limit our “Froissart-like” bound is stable

- masses of nuclei, baryons, and non-Goldstone mesons are expected to change only by a few MeV
- the presence of massless pions can at most make some particle unstable, not the other way around
- \(\Omega\) expected to remain stable and with approximately the same mass, so it is expected to still be the dominant particle

[MG and E. Meggiolaro, in preparation]
Conclusions and Outlook

Main results:

- $\sigma_{\text{tot}} \sim B \log^2 s$, if higher-spin stable states exist
- “Froissart-like” bound $B \leq \frac{4\pi}{\mu^2}$, $\frac{1}{\mu} = \max s^{(a)} \frac{1}{m^{(a)}}$ determined by the spectrum with $\mu = 2m_\Omega$
  - more restrictive and stable in the chiral limit
- Under the “black disk assumption”, $\sigma_{\text{tot}}$ is universal and entirely determined from the spectrum, $B_{\text{th}} = \frac{2\pi}{\mu^2} \sim B_{\text{exp}}$ within $20 \div 25$
  - first subleading term in $\sigma_{\text{tot}}$ is $\propto \log s \cdot \log \log s$ and universal
  - universal black-disk amplitude for elastic scattering

Open issues:

- Why large unquenching effects?
- What relation with Regge theory?
References

- H. Verlinde and E. Verlinde, hep-th/9302104
- M. Giordano and R. Peschanski, JHEP 05 (2010) 037
- M. Giordano, E. Meggiolaro and N. Moretti, JHEP 09 (2012) 031
- E. Gregory at al., JHEP 10 (2012) 170
Subleading Terms

\[ \sigma_{\text{tot}} \sim 2\pi(\tilde{s} - 1)^2 \frac{1}{\tilde{m}^2} [1 - \text{Re} \langle \langle g_\infty(\nu_1, \nu_2) \rangle \rangle] \eta^2 + O(\eta) \]

\[ \eta = \chi(\tilde{s} - 1) - \frac{1}{2} \log \chi \]

If extra inverse powers \((bm)^{-\frac{\lambda}{2}}\) are present in \(W_\alpha, \overline{W}_\alpha\)

\[ \eta = \chi(\tilde{s} - 1) - \frac{1 + \lambda}{2} \log \chi \]

First subleading term in \(\sigma_{\text{tot}}\) is \(\propto \chi \log \chi \sim \log s \cdot \log \log s\)
If \( g_\infty (\nu_1, \nu_2) = 0 \)

\[
\mathcal{M}(s, t) \underset{s \to \infty}{\sim} 4\pi i s \frac{\eta}{q \tilde{m}} J_1 \left( \frac{q \eta}{\tilde{m}} \right) = i s \sigma_{\text{tot}} \frac{2J_1(\varrho)}{\varrho}
\]

\( \varrho = \frac{\sqrt{-t \eta}}{\tilde{m}} \approx \frac{\sqrt{-t \log \frac{s}{m_1 m_2}}}{\tilde{m}} \)

- Black disk amplitude (purely imaginary):
  - total elastic cross section \( \frac{\sigma_{\text{el}}}{\sigma_{\text{tot}}} = \frac{1}{2} \)
  - \( B \)-slope \( B \equiv \frac{d}{dt} \log \frac{d\sigma_{\text{el}}}{dt} \bigg|_{t=0} \) satisfies \( \frac{8\pi B}{\sigma_{\text{tot}}} = 1 \)
  - zero at \( t_0 \), \( \frac{|t_0|\sigma_{\text{tot}}}{2\pi x_0^2} = 1 \), \( x_0 \approx 3.8 \approx \) dip in \( \frac{d\sigma_{\text{el}}}{dt} \bigg|_{t_{\text{dip}}} \) \[ \frac{|t_{\text{dip}}|\sigma_{\text{tot}}}{2\pi x_0^2} \bigg|_{\exp} \) well above 1

\[ \left[ \frac{\sigma_{\text{el}}}{\sigma_{\text{tot}}} \right]_{\exp, \sqrt{s}=7 \text{ TeV}} \approx 0.26 \]
\[ \left[ \frac{8\pi B}{\sigma_{\text{tot}}} \right]_{\exp, \sqrt{s}=7 \text{ TeV}} \approx 1.97 \]

\[ \text{[Csörgő, Nemes (2014)]} \]

\( \mathcal{M}(s, t)/\mathcal{M}(s, 0) = 2J_1(\varrho)/\varrho \) depends on \( t \) only through \( \varrho \)
- universal function of \( \varrho \)
- entire of order 1 in \( \varrho \)

\[ \text{[Auberson, Kinoshita, Martin (1971)]} \]
Main assumption: possibility to interchange the order of $\sum_\alpha$ and $\theta \to -i\chi$

Partially justified by the short-range ($a$) nature of strong interactions

LSZ framework: $W_\alpha, \overline{W}_\alpha \leftrightarrow \int [d^4x e^{ip\cdot x}]^n \langle 0| T\{\hat{\mathcal{W}} \Phi_1(x_1) \ldots \Phi_n(x_n)\}|0\rangle$

For most of the configurations $|x_i - x_j|/a \gg 1$, mutual interactions negligible, interaction with Wilson loop only local

$$W_\alpha(\{\vec{p}\}, \{s_3\}; \nu_1) \simeq \prod_{a,i} \frac{\langle 0|\hat{\mathcal{W}}_0(\nu_1)|\vec{p}^{(a)}_i, s_3^{(a)}_i\rangle}{\langle 0|\hat{\mathcal{W}}_0(\nu_1)|0\rangle} = \prod_{a,i} W_a(\vec{p}^{(a)}_i, s_3^{(a)}_i; \nu_1)$$

$$\tilde{C}_E \simeq \exp \left\{ \sum_a \sum_{s_3} e^{i\theta s_3} \int d\Omega_a \ e^{-b\varepsilon^{(a)}} \ W_a(\vec{p}_{a \frac{\theta}{2}}, s_3; \nu_1) \overline{W}_a(\vec{p}_{a - \frac{\theta}{2}}, s_3; \nu_2) \right\} - 1$$

Resummation and analytic continuation commute
Lattice calculations give “true” prediction of QCD (within errors) ⇒ test analytic NP calculations

Numerical predictions and fits of model functions of SVM (top) and ILM (bottom) to lattice data

Lattice setup:

- Wilson action for $SU(3)$ gauge theory (*quenched* QCD)
- $16^4$ hypercubic lattice, $a \simeq 0.1$ fm
- longest available loops ($L \simeq 8$)
- $\cot \theta = 0, \pm 1/2, \pm 1, \pm 2$
- $|\vec{r}_{1,2\perp}| = 1a, |\vec{d}_\perp| = 0, 1, 2a$
- “zzz”: $\vec{d}_\perp \parallel \vec{r}_{1\perp} \parallel \vec{r}_{2\perp}$
- “zyy”: $\vec{d}_\perp \perp \vec{r}_{1\perp} \parallel \vec{r}_{2\perp}$
- “ave”: average over orientations
Rising Cross Sections From the Lattice

Parameterisation: \( C_E = e^{K_E} - 1 \)
\[
K_E = \sum_i f_i(\theta) g_i(\vec{b}_\perp, \nu_1, \nu_2)
\]

Unitarity constraint: \( \text{Re } K_M \leq 0 \)
\[
[K_M(\chi) = K_E(\theta \rightarrow -i\chi)]
\]

At large \( b \), \( K_E, K_M \sim (\sum_j) e^{-\mu_j b} \)

If \( K_M \sim \chi^p e^{n\chi} e^{-\mu b} \sim (\log s)^p s^n e^{-\mu b} \)
\[
\sigma_{\text{tot}}^{(hh)} \sim B \log^2 s
\]

with \( B = \frac{2\pi n^2}{\mu^2} \) universal

Estimate of \( B \) fairly agrees with \( B_{\text{exp}} \)
(although quenched and with rather large errors)

Where does this come from?

\[
K_E = \frac{K_1}{\sin \theta} + K_2(\frac{\pi}{2} - \theta)^3 \cos \theta
\]
More on the Scaling Function

Two possible kinds of dominant particle

(1) Self-conjugate boson \((Q = B = S = \ldots = 0)\)

(2) Boson with nonzero charges, or fermion

\[
\tilde{C}_M \to g(z; \nu_1, \nu_2) - 1 = \left\{ \begin{array}{ll}
\sum_{n=1}^{\infty} \frac{1}{n!} C^0_n(\nu_1, \nu_2) z^n \\
\sum_{n=1}^{\infty} \frac{1}{(2n)!} C_n(\nu_1, \nu_2) z^{2n}
\end{array} \right. \quad (1)
\]
Universal “Froissart-like” Total Cross Section

\[ \sigma_{\text{tot}} \simeq \frac{4\pi}{\tilde{m}^2} \text{Re} \langle J \rangle \]

\[ J = \int_0^{e^\eta} \frac{dz}{z} \log \frac{e^\eta}{z} \left[ 1 - g(z; \nu_1, \nu_2) \right] \]

\[ e^\eta = \frac{e^{\chi(\tilde{s} - 1)}}{\sqrt{\chi}} \]

\[ \eta = \chi(\tilde{s} - 1) - \frac{1}{2} \log \chi \]

Look for \( \mathcal{O}(\eta^2) \) terms in \( J = J_1 - J_2 + J_3 \)

\[ J_1 = \int_{1}^{e^\eta} \frac{dz}{z} \log \frac{e^\eta}{z} = \frac{1}{2} \eta^2 \]

\[ J_2 = \int_{1}^{e^\eta} \frac{dz}{z} \log \frac{e^\eta}{z} g(z; \nu_1, \nu_2) = \frac{1}{2} \eta^2 g_\infty(\nu_1, \nu_2) + \mathcal{O}(\eta) \]

\[ J_3 = \int_{0}^{1} \frac{dz}{z} \log \frac{e^\eta}{z} \left[ 1 - g(z; \nu_1, \nu_2) \right] = \mathcal{O}(\eta) \]