Hadronic total cross section, Wilson loop correlators and the QCD spectrum

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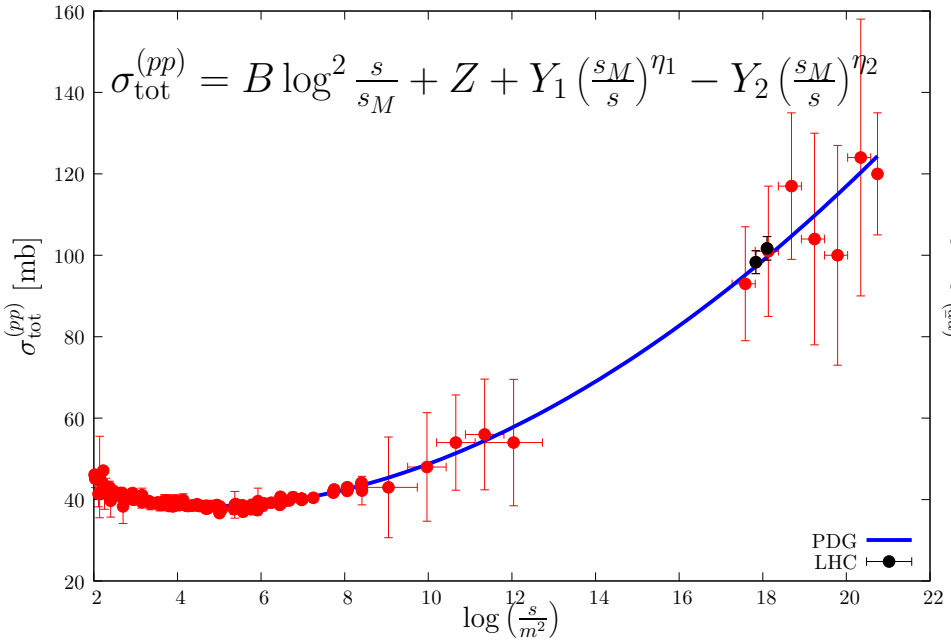
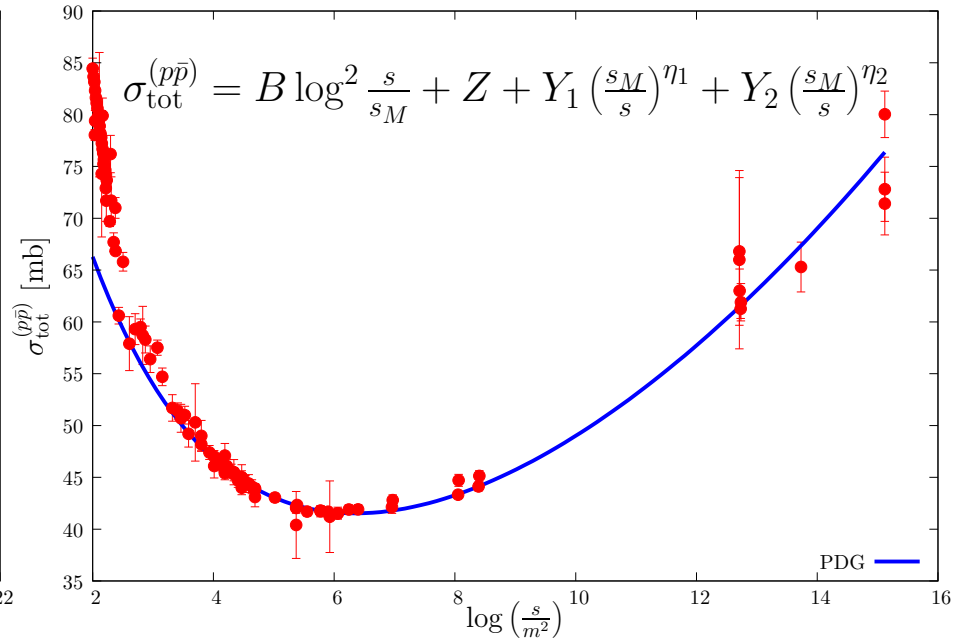
Rising Total Cross Sections





[PDG (2013)][TOTEM (2013)]

$$\sigma_{tot}^{(hh)}(s) \sim B \log^2 s$$

$B \simeq 0.3\,\mathrm{mb}$ **universal**, independent of the colliding hadrons

Consistent with Froissart bound (unitarity + mass gap) [Froissart (1961)]

$$\sigma_{tot}^{(hh)}(s) \leq \frac{\pi}{m_\pi^2} \log^2 \frac{s}{s_0}$$

Soft High-Energy Scattering and Total Cross Sections

Total cross sections related to forward elastic amplitudes via optical theorem

$$\sigma_{\rm tot} \underset{s\to\infty}{\simeq} \frac{1}{s}\,{\rm Im}\,\mathcal{M}(s,t=0)$$

Soft high-energy hadron-hadron scattering: $s \to \infty\,,\ |t| \leq 1\,{\rm GeV}^2$
One of the oldest unsolved problems of strong interactions

Impact parameter representation $[t = -\vec{q}_\perp^{\,2}]$

$$\mathcal{M}(s,t) = i\ 2s \int d^2\vec{b}_\perp e^{i\vec{q}_\perp\cdot\vec{b}_\perp} A(s,\vec{b}_\perp)$$

$$\sigma_{\rm tot} = 2{\rm Re} \int d^2\vec{b}_\perp\, A(s,\vec{b}_\perp)$$

Unitarity: $|A(s,\vec{b}_\perp) - 1| \leq 1$

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Impact parameter representation for unpolarised scattering $[t = -\vec{q}_\perp^{\,2}]$

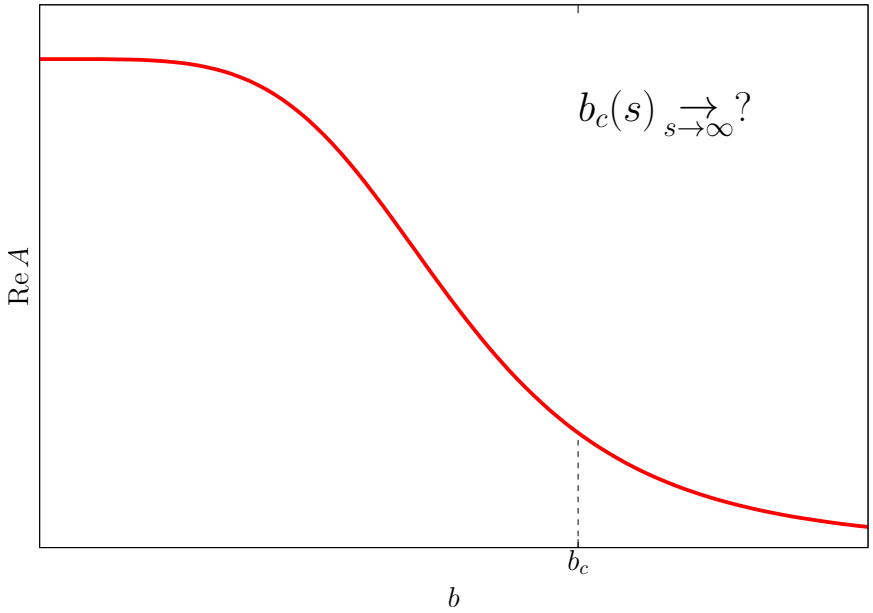
$$\mathcal{M}(s,t) = i\ 4\pi s \int_0^\infty dbb\, J_0(bq) A(s,b)$$

$$\sigma_{\rm tot} = 4\pi {\rm Re} \int_0^\infty dbb\, A(s,b)$$

Unitarity: $|A(s,b) - 1| \leq 1$

How to Obtain a Rising Total Cross Section

Typical shape: $A \to 0$ at large b, for $b > b_c(s)$ the amplitude is "negligible"



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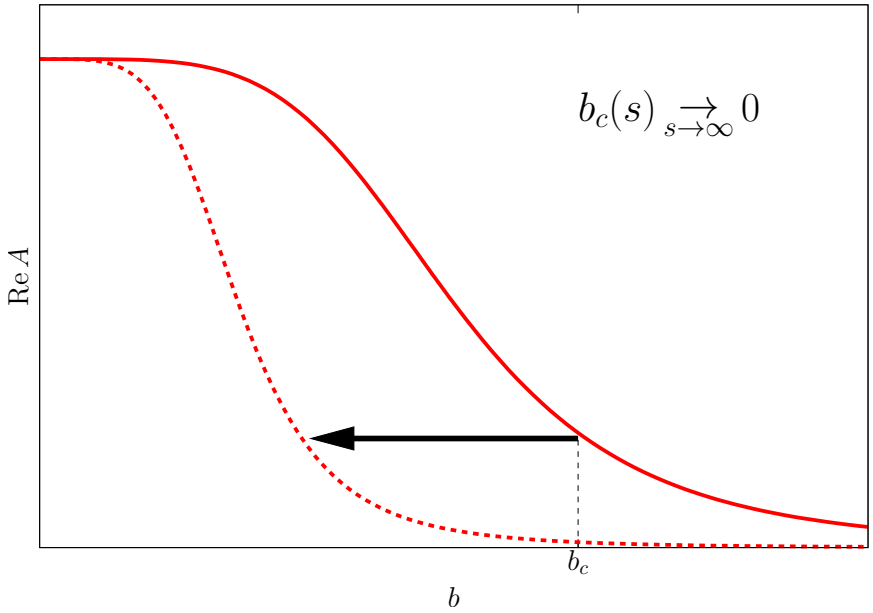

$\sigma_{\rm tot} \sim b_c(s)^2$: how does b_c change with s? $b_c(s) \to 0$

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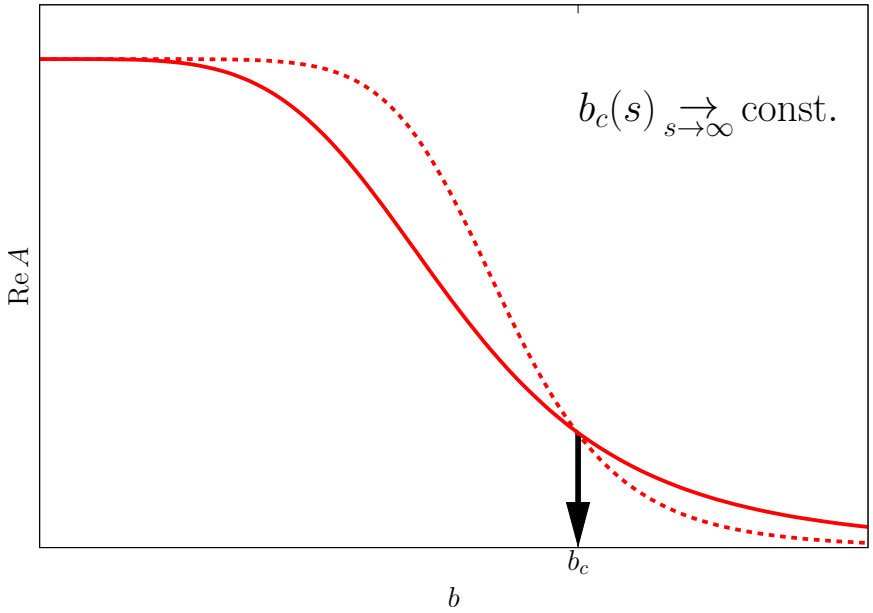

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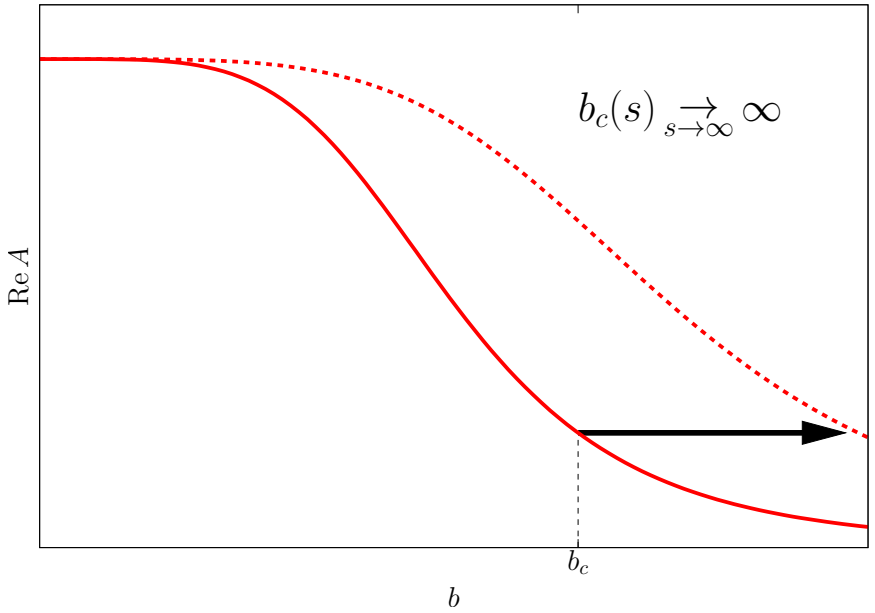

$\sigma_{\rm tot} \sim b_c(s)^2$: how does b_c change with s? $b_c(s) \to \infty$

$$\sigma_{\rm tot} = 4\pi b_c(s)^2 \,\mathrm{Re} \int_0^\infty dx\, x\, A(s, b_c(s)x) \to 4\pi b_c(s)^2 C$$

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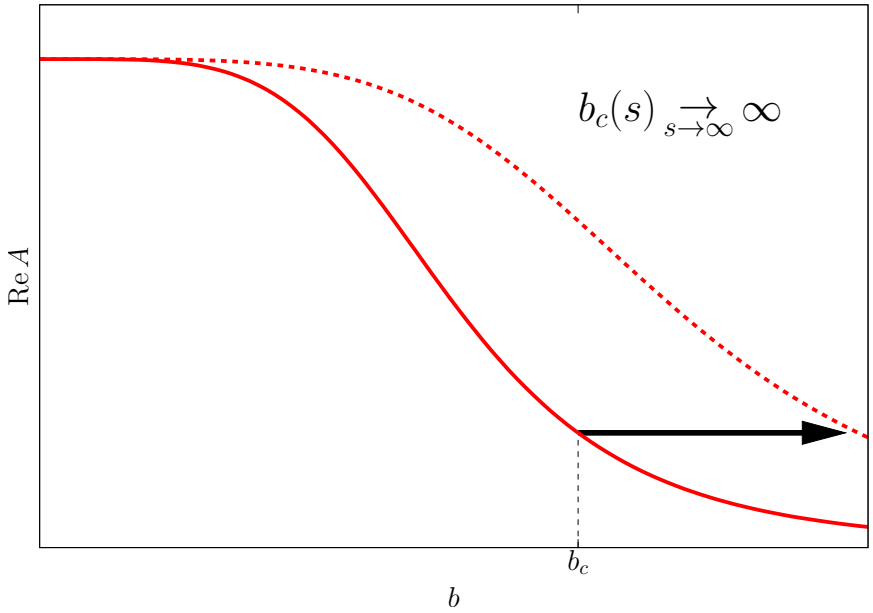

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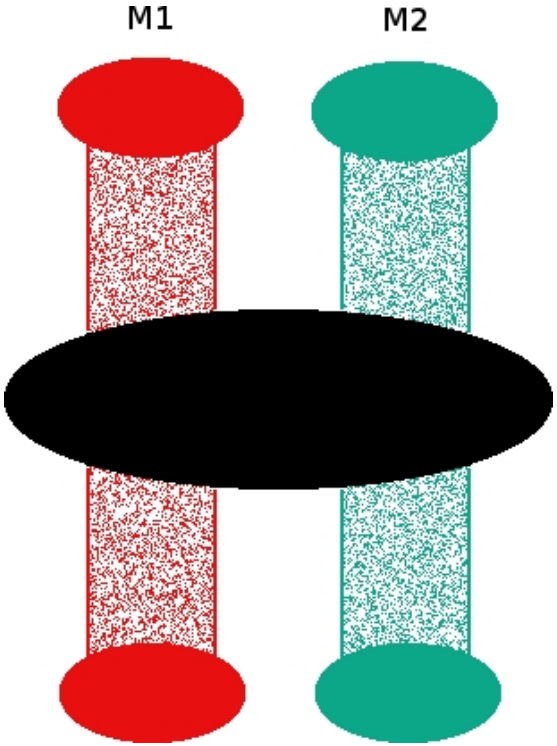
- $b_c(s)$ gives the energy dependence
- large b ($\gg m^{-1}$) is relevant

Soft High-Energy Scattering and QCD

QCD: fundamental theory, should explain the rise of total cross sections

$|t| \lesssim 1\mathrm{GeV}^2$, PT not fully reliable $\rightarrow$ NP approach [Nachtmann (1991)]



1. Partonic description of hadrons over a small time–window ($\sim 2\mathrm{fm}$)
2. Partons do not split or annihilate, treated as in/out states of a scattering process
3. Lightlike trajectories approx. unchanged in the process, only soft gluon exchange
4. Hadronic amplitude after folding with hadronic wave function

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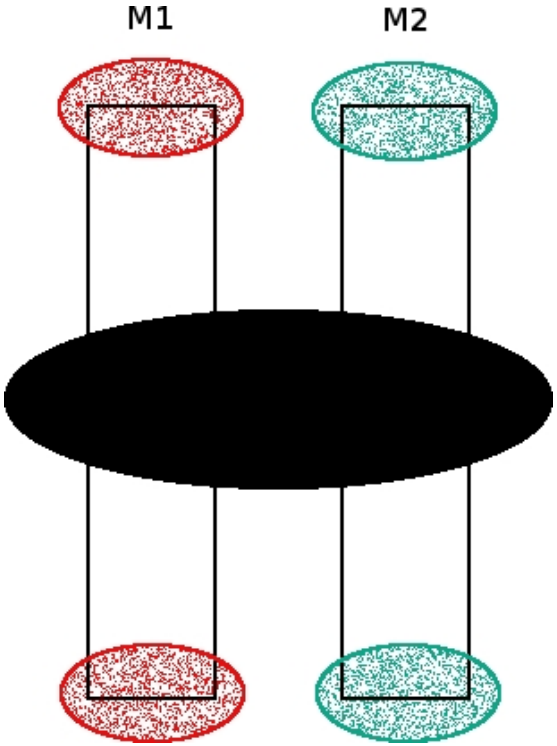

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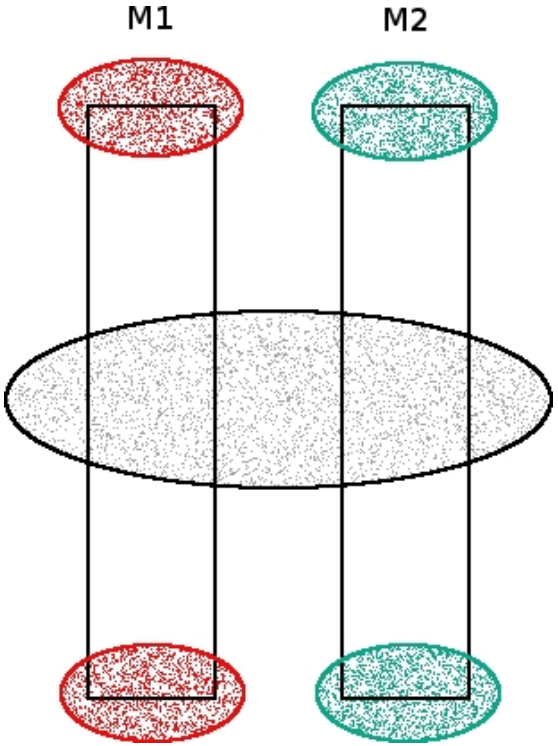

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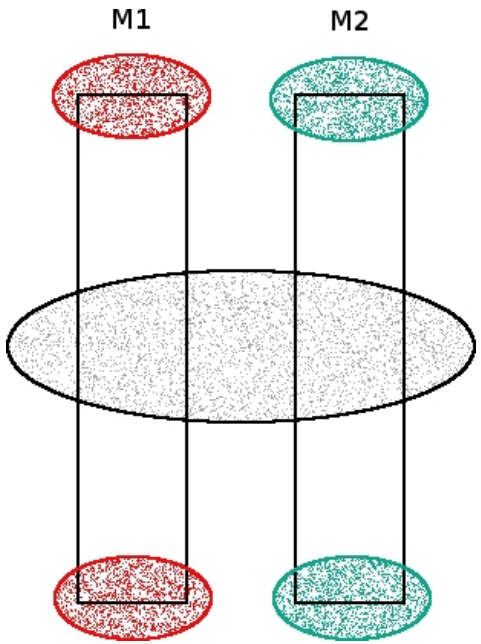
Nonperturbative Approach

Partonic scattering amplitudes from the correlation function of infinite lightlike Wilson lines [Nachtmann (1991)]

To avoid IR divergences $\rightarrow$ hadronic amplitudes

- mesons as wave packets of transverse colourless dipoles
- dipole scattering amplitudes from the correlation function of infinite lightlike Wilson loops [Dosch *et al.* (1996)]



Intermediate regularisation: finite hyperbolic angle χ and length $2T$ [Verlinde, Verlinde (1993)]

Extends to baryon-baryon scattering adopting a quark-diquark description[Rueter, Dosch (1996)]

Meson-Meson (Dipole-Dipole) Elastic Scattering

Elastic meson-meson from dipole-dipole scattering [Dosch *et al.* (1996)]

$$A(s, \vec{b}_\perp) = \langle\langle A^{(dd)}(s, \vec{b}_\perp; \nu_1, \nu_2) \rangle\rangle$$

$\nu_i = (f_i, \vec{R}_{i\perp})$, f_i longitudinal momentum fraction, $\vec{R}_{i\perp}$ transverse size

$\langle\langle \ldots \rangle\rangle$: average over $\nu_{1,2}$ with mesonic wave functions, $\langle\langle 1 \rangle\rangle = 1$

dd scattering amplitude in *b*-space $\leftrightarrow$ Wilson-loop correlation function

$$-A^{(dd)}(s, \vec{b}_\perp; \nu_1, \nu_2) = \lim_{\chi \to \infty} \mathcal{C}_M(\chi; \vec{b}_\perp, \nu_1, \nu_2) \qquad \chi \underset{s \to \infty}{\simeq} \log \frac{s}{m^2}$$

$$\mathcal{G}_M(\chi; T; \vec{b}_\perp, \nu_1, \nu_2) \equiv \frac{\langle \mathcal{W}_{\mathcal{C}_1} \mathcal{W}_{\mathcal{C}_2} \rangle}{\langle \mathcal{W}_{\mathcal{C}_1} \rangle \langle \mathcal{W}_{\mathcal{C}_2} \rangle} - 1 \qquad \mathcal{C}_M \equiv \lim_{T \to \infty} \mathcal{G}_M$$

$\langle \ldots \rangle$: expectation value in the functional integral formalism

Wilson Loop Correlation Function

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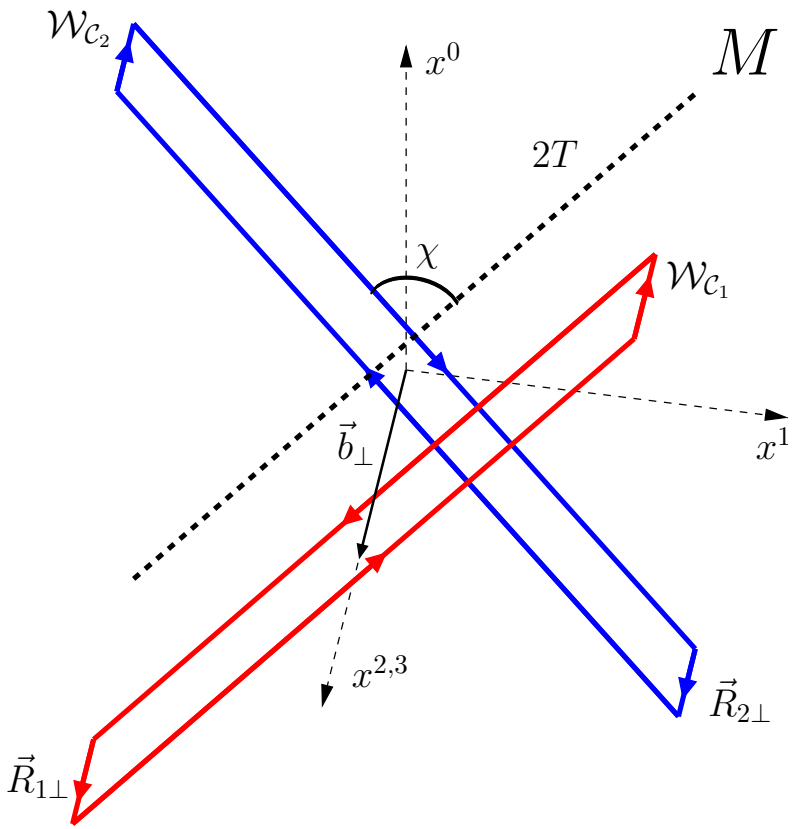
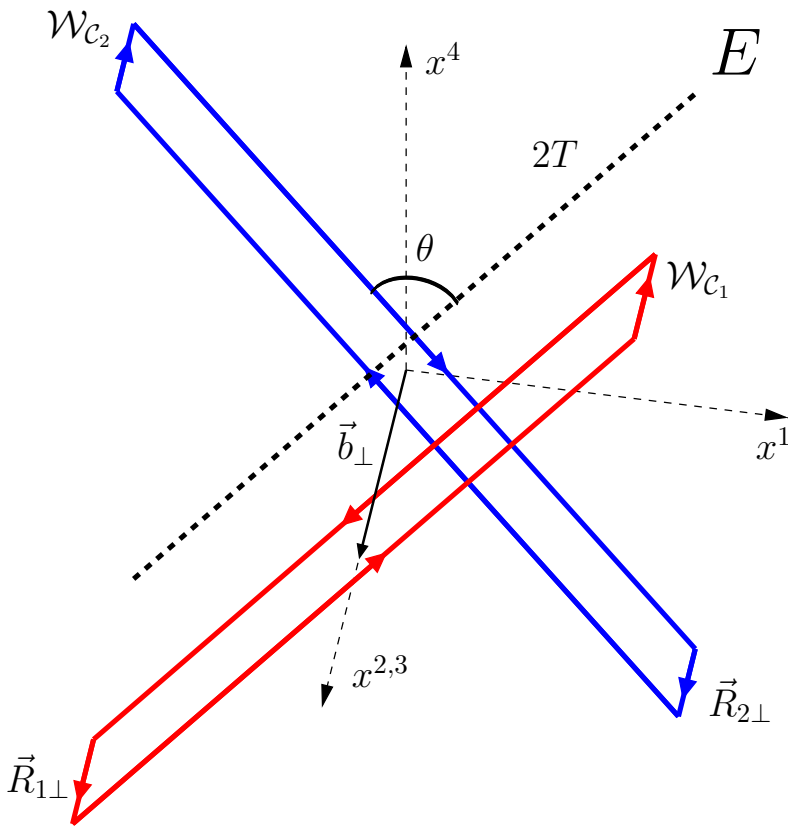


Wilson Loop Correlation Function

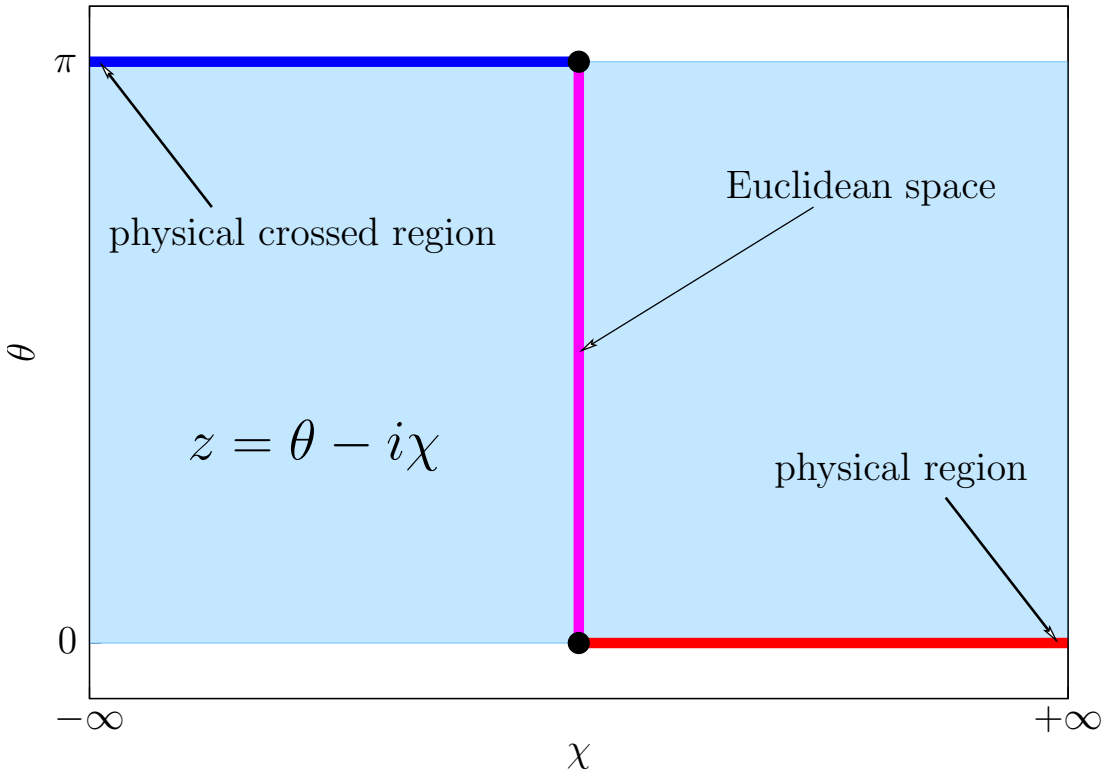
NP techniques available in Euclidean space $\Rightarrow$ Euclidean formulation

[Meggiolaro (1997), Meggiolaro (2005)]

$$\mathcal{G}_E(\theta; T; \vec{b}_\perp, \nu_1, \nu_2) \equiv \frac{\langle \mathcal{W}_{\mathcal{C}_1} \mathcal{W}_{\mathcal{C}_2} \rangle}{\langle \mathcal{W}_{\mathcal{C}_1} \rangle \langle \mathcal{W}_{\mathcal{C}_2} \rangle} - 1\,, \qquad \mathcal{C}_E \equiv \lim_{T \to \infty} \mathcal{G}_E$$



Analytic Continuation to Euclidean Space



Analytic continuation relations [Meggiolaro (2005), MG, Meggiolaro (2009)]

$$\mathcal{C}_M(\chi) = \mathcal{C}_E(\theta \to -i\chi)$$

AC + Euclidean symmetries $\Rightarrow$ crossing relations [MG, Meggiolaro (2006)]

$$\mathcal{C}_M(i\pi - \chi; \vec{R}_{1\perp}, \vec{R}_{2\perp}) = \mathcal{C}_M(\chi; \vec{R}_{1\perp}, -\vec{R}_{2\perp})$$

Nonperturbative Models

Euclidean formulation opens the way to NP techniques:

- Stochastic Vacuum Model [Berger, Nachtmann (1999), Shoshi *et al.* (2003)]
- Instanton Liquid Model [Shuryak, Zahed (2000), MG, Meggiolaro (2010)]
- AdS/CFT Correspondence [Janik, Peschanski (2000a,b), MG, Peschanski (2010)]
- Lattice Gauge Theory [MG, Meggiolaro (2008), MG, Meggiolaro (2010)]

Stochastic Vacuum Model (SVM)	$\mathcal{C}_E = \frac{2}{3} e^{-\frac{1}{3}\cot\theta\, K_{\rm SVM}} + \frac{1}{3} e^{\frac{2}{3}\cot\theta\, K_{\rm SVM}} - 1$
Instanton Liquid Model (ILM)	$\mathcal{C}_E = \frac{K_{\rm ILM}}{\sin\theta}$
Perturbation Theory (PT)	$\mathcal{C}_E = K_{\rm PT} \cot^2\theta$
ILM + PT (ILMp)	$\mathcal{C}_E = \frac{K_{\rm ILMp}}{\sin\theta} + K'_{\rm ILMp} (\cot\theta)^2$
AdS/CFT correspondence	$\mathcal{C}_E = e^{\frac{K_{\rm AdS}}{\sin\theta} + K'_{\rm AdS}\cot\theta + K''_{\rm AdS}\cos\theta\cot\theta} - 1$

NP Models, Lattice Results and Rising Cross Sections

Lattice calculations give "true" prediction of QCD (within errors) $\Rightarrow$ test analytic NP calculations

Are the analytic NP calculations compatible with the lattice results?

- SVM/ILM do not match/fit well the data and $\sigma_{tot}^{\mathrm{SVM,ILM}} \underset{s\to\infty}{\to} \mathrm{const.}$
- ILM+PT gives improved best fits but $\sigma_{tot}^{\mathrm{ILMp}} \underset{s\to\infty}{\to} \mathrm{const.}$
- AdS/CFT: $\sigma_{tot} \propto s^{\frac{1}{3}}$ but for onium-onium scattering in $\mathcal{N} = 4$ SYM

[MG, Peschanski (2010)]

Are the lattice results compatible with rising total cross sections?

- More general fits, but care is needed because of the AC
- Constrain admissible fitting functions with physical requirements (unitarity, crossing symmetry,...)
- Parameterisations fitting well the data and leading to rising total cross sections exist [MG, Meggiolaro, Moretti (2012)]

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Summary and Questions

- $\sigma_{\rm tot} \sim$ large-b behaviour of elastic scattering amplitudes in impact-parameter space $A(s,b)$
- QCD at large-s and small-t: $A(s,b) \sim$ Wilson loop correlation function
- Analytic models fail to reproduce the lattice data and to capture the rising behaviour of $\sigma_{\rm tot}$
- Lattice data compatible with rising behaviour, but large arbitrariness in the parameterisations

1. What are the large-s and large-b behaviour of $A(s,b)$?
2. What sets the physical scale in $\sigma_{\rm tot}$?
3. How does $\sigma_{\rm tot}$ relate to the hadronic spectrum?

Relating Total Cross Sections and the QCD Spectrum

How to extract θ and b dependencies?

Basic idea: insert a complete set of states between the Wilson loops

$$\langle 0|\mathcal{O}_1(t)\mathcal{O}_2(0)|0\rangle = \sum_n e^{-E_n t}\langle 0|\mathcal{O}_1(0)|n\rangle\langle n|\mathcal{O}_2(0)|0\rangle$$

Complications: nonlocal operators, nontrivial angular dependence

Use asymptotic states with simple transformation properties

$$|\alpha\rangle = |\{n_a(\alpha)\}, \{\vec{p}\}, \{s_3\}\rangle$$

$\{n_a(\alpha)\}$: particle content, $\{\vec{p}\}$: momenta, $\{s_3\}$: 3rd component of spin

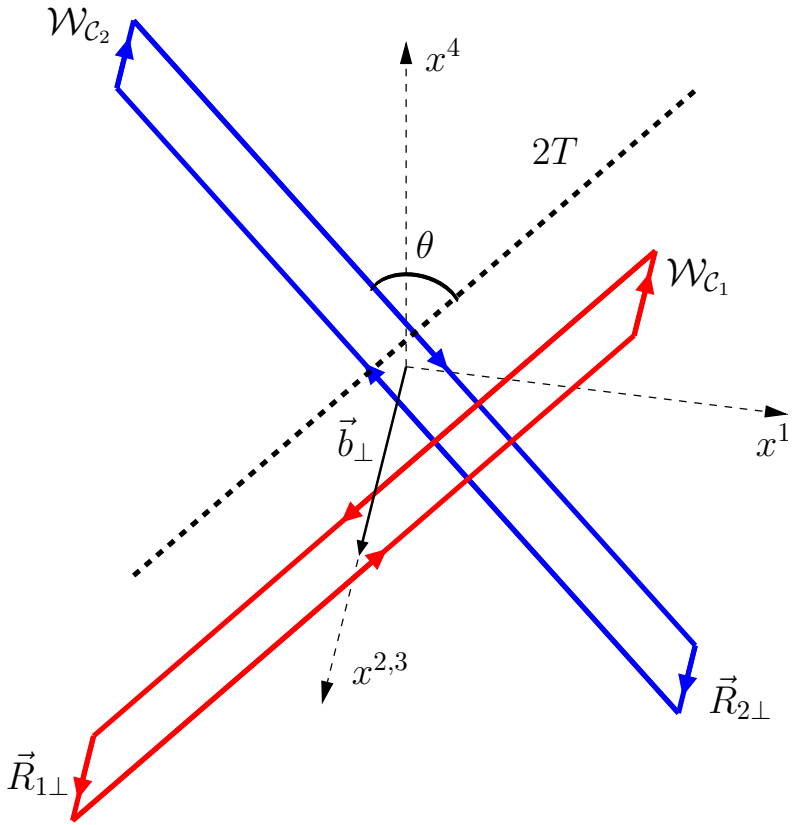
$$\sum_n |n\rangle\langle n| = \sum_\alpha \mathcal{P}_\alpha \sum_{\{s_3\}_\alpha} \int d\Omega_\alpha \, |\alpha\rangle\langle\alpha|$$

$\mathcal{P}_\alpha = \frac{1}{\prod_a n_a(\alpha)!}$ symmetry factor, $d\Omega_\alpha$ phase space measure

Sketch of Derivation 1

Rotate Euclidean time along the impact parameter (equivalent description)

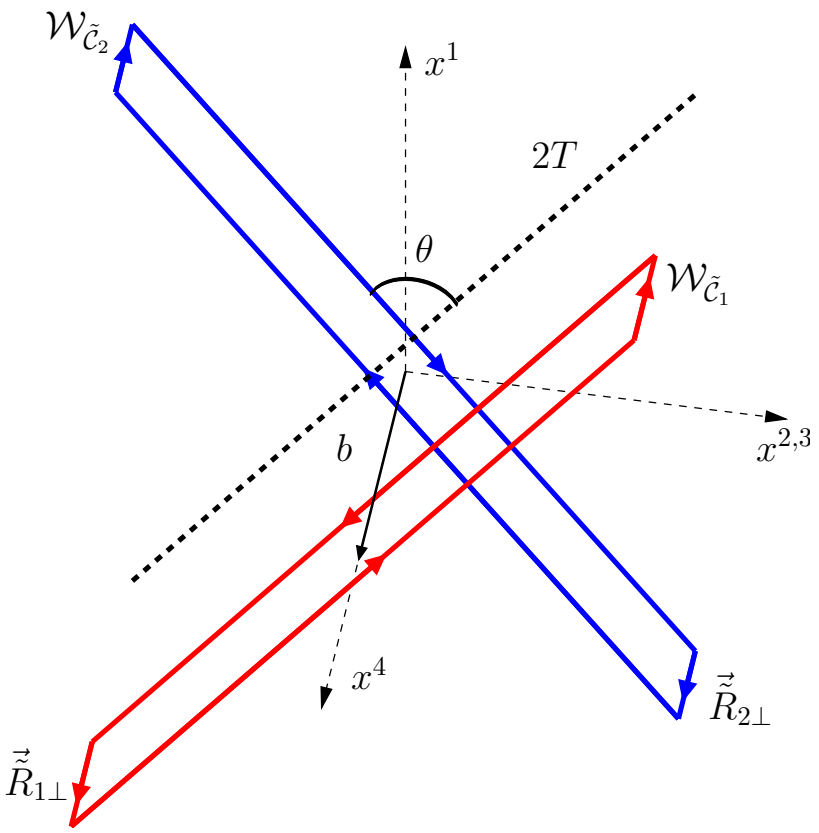
$$\mathcal{G}_E(\theta; T; \vec{b}_\perp, \nu_1, \nu_2) \equiv \frac{\langle \mathcal{W}_{\mathcal{C}_1} \mathcal{W}_{\mathcal{C}_2} \rangle}{\langle \mathcal{W}_{\mathcal{C}_1} \rangle \langle \mathcal{W}_{\mathcal{C}_2} \rangle} - 1$$



Sketch of Derivation 1

Rotate Euclidean time along the impact parameter (equivalent description)

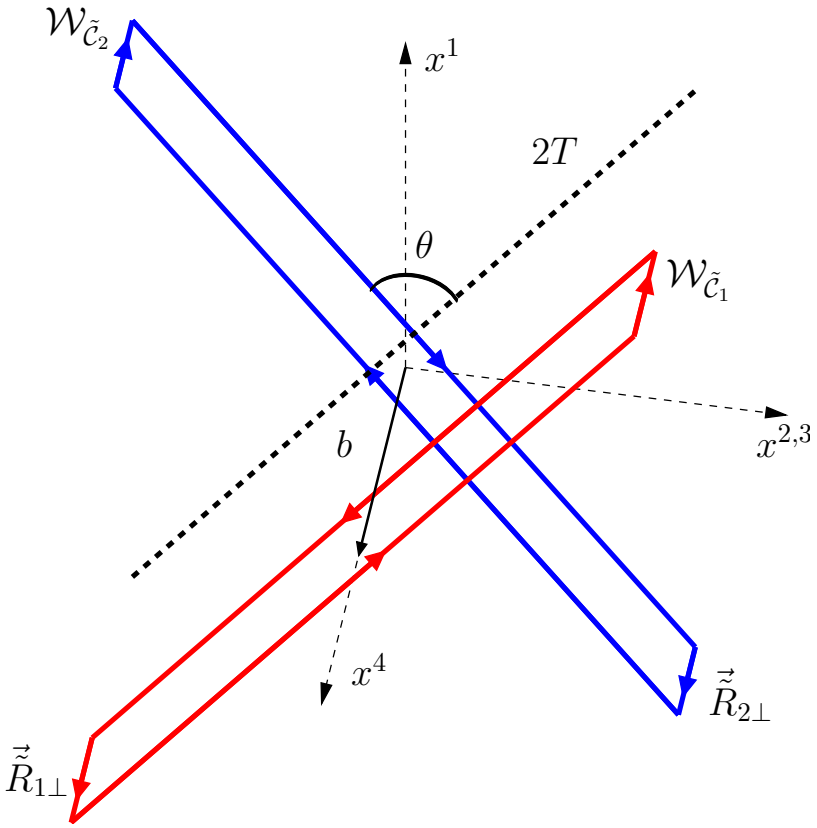
$$\tilde{\mathcal{G}}_E(\theta; T; b; \nu_1, \nu_2) \equiv \frac{\langle \mathcal{W}_{\tilde{\mathcal{C}}_1} \mathcal{W}_{\tilde{\mathcal{C}}_2} \rangle}{\langle \mathcal{W}_{\tilde{\mathcal{C}}_1} \rangle \langle \mathcal{W}_{\tilde{\mathcal{C}}_2} \rangle} - 1$$



Sketch of Derivation 2

Use Wilson loop *operators*

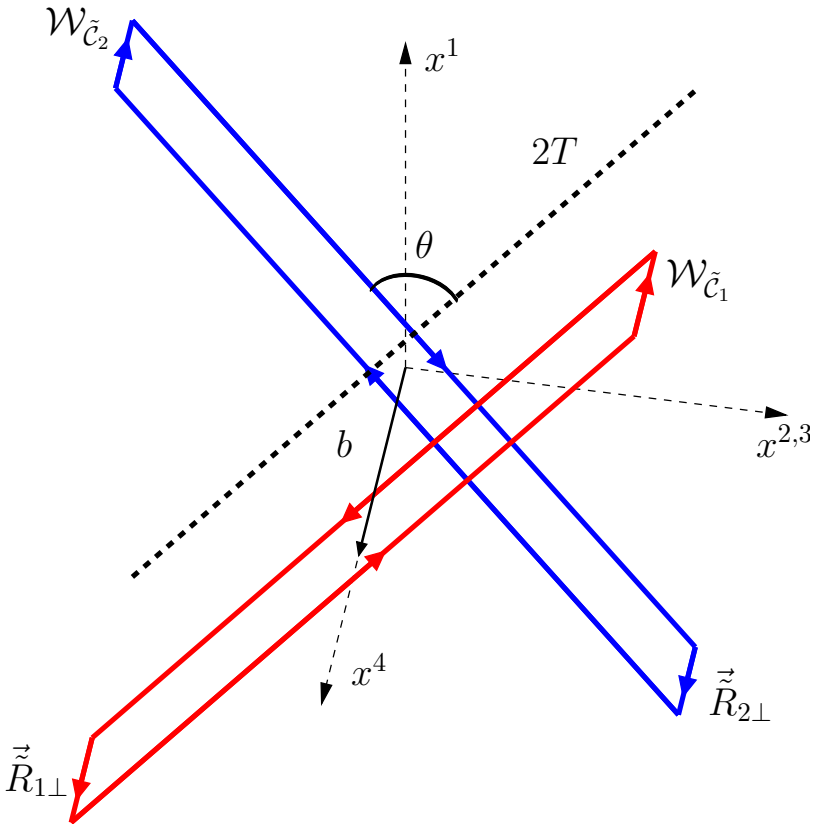
$$\tilde{\mathcal{G}}_E(\theta; T; b; \nu_1, \nu_2) \equiv \frac{\langle \mathcal{W}_{\tilde{\mathcal{C}}_1} \mathcal{W}_{\tilde{\mathcal{C}}_2} \rangle}{\langle \mathcal{W}_{\tilde{\mathcal{C}}_1} \rangle \langle \mathcal{W}_{\tilde{\mathcal{C}}_2} \rangle} - 1 = \frac{\langle 0 | T\{\hat{\mathcal{W}}_{\tilde{\mathcal{C}}_1} \hat{\mathcal{W}}_{\tilde{\mathcal{C}}_2}\} | 0 \rangle}{\langle 0 | \hat{\mathcal{W}}_{\tilde{\mathcal{C}}_1} | 0 \rangle \langle 0 | \hat{\mathcal{W}}_{\tilde{\mathcal{C}}_2} | 0 \rangle} - 1$$



Sketch of Derivation 3

Consider loops with no temporal overlap ($b > b_0$)

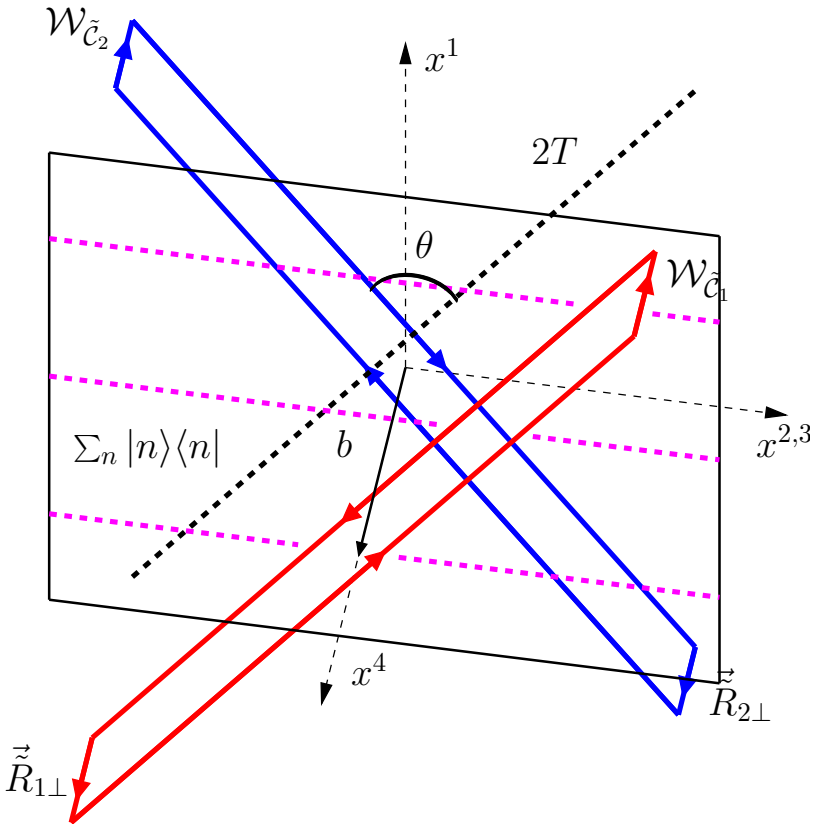
$$\tilde{\mathcal{G}}_E(\theta; T; b; \nu_1, \nu_2) \equiv \frac{\langle \mathcal{W}_{\tilde{\mathcal{C}}_1} \mathcal{W}_{\tilde{\mathcal{C}}_2} \rangle}{\langle \mathcal{W}_{\tilde{\mathcal{C}}_1} \rangle \langle \mathcal{W}_{\tilde{\mathcal{C}}_2} \rangle} - 1 = \frac{\langle 0 | \hat{\mathcal{W}}_{\tilde{\mathcal{C}}_1} \hat{\mathcal{W}}_{\tilde{\mathcal{C}}_2} | 0 \rangle}{\langle 0 | \hat{\mathcal{W}}_{\tilde{\mathcal{C}}_1} | 0 \rangle \langle 0 | \hat{\mathcal{W}}_{\tilde{\mathcal{C}}_2} | 0 \rangle} - 1$$



Sketch of Derivation 4

Insert a complete set of states

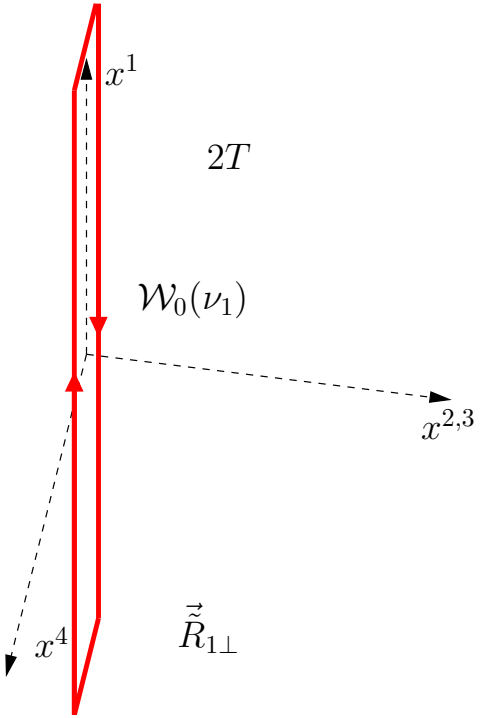
$$\tilde{\mathcal{G}}_E(\theta; T; b; \nu_1, \nu_2) = \frac{\langle 0|\hat{\mathcal{W}}_{\tilde{\mathcal{C}}_1}\hat{\mathcal{W}}_{\tilde{\mathcal{C}}_2}|0\rangle}{\langle 0|\hat{\mathcal{W}}_{\tilde{\mathcal{C}}_1}|0\rangle\langle 0|\hat{\mathcal{W}}_{\tilde{\mathcal{C}}_2}|0\rangle} - 1 = \sum_{n\neq 0} \frac{\langle 0|\hat{\mathcal{W}}_{\tilde{\mathcal{C}}_1}|n\rangle}{\langle 0|\hat{\mathcal{W}}_{\tilde{\mathcal{C}}_1}|0\rangle} \frac{\langle n|\hat{\mathcal{W}}_{\tilde{\mathcal{C}}_2}|0\rangle}{\langle 0|\hat{\mathcal{W}}_{\tilde{\mathcal{C}}_2}|0\rangle}$$

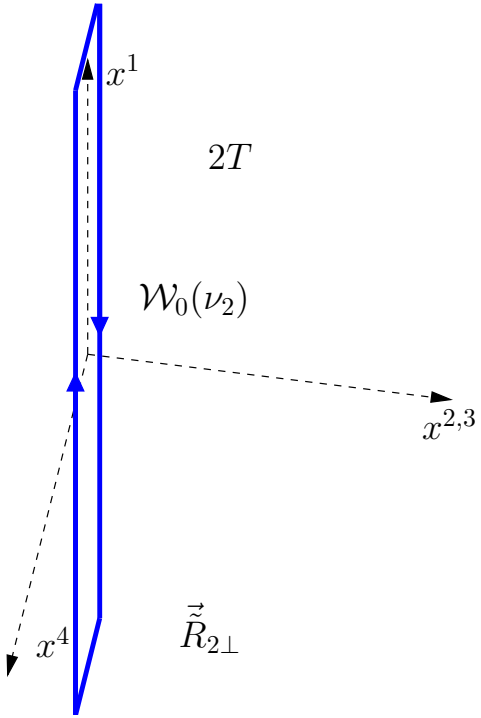


Sketch of Derivation 5

Rotate around 3-axis and translate centres to the origin, $|n_{\pm\frac{\theta}{2}}\rangle = e^{\pm i\hat{J}_3\frac{\theta}{2}}|n\rangle$

$$\tilde{\mathcal{G}}_E(\theta;T;b;\nu_1,\nu_2) = \sum_{n\neq 0} e^{-bE_n} e^{i\theta S_{3n}} \frac{\langle 0|\hat{\mathcal{W}}_0(\nu_1)|n_{\frac{\theta}{2}}\rangle}{\langle 0|\hat{\mathcal{W}}_0(\nu_1)|0\rangle} \frac{\langle n_{-\frac{\theta}{2}}|\hat{\mathcal{W}}_0(\nu_2)|0\rangle}{\langle 0|\hat{\mathcal{W}}_0(\nu_2)|0\rangle}$$





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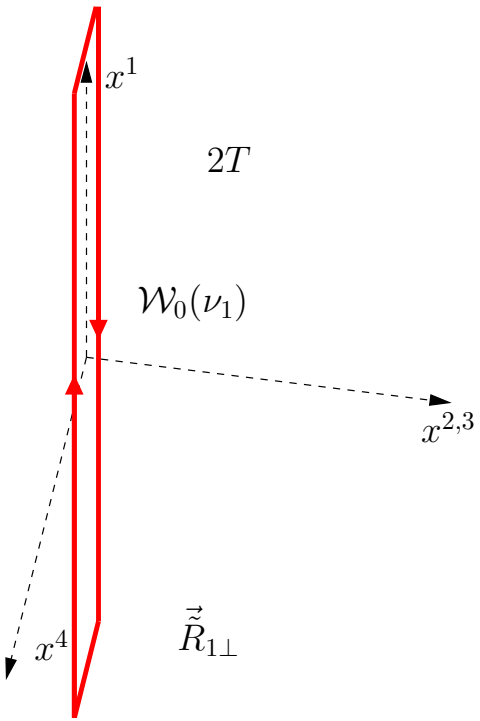
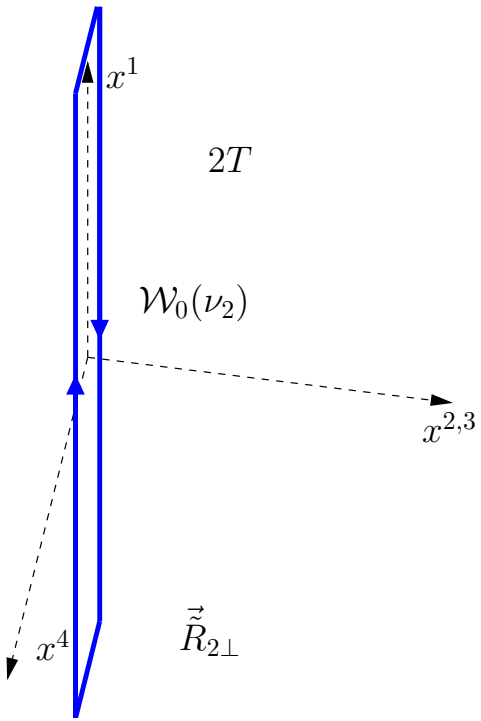
$$\tilde{\mathcal{G}}_E(\theta;T;b;\nu_1,\nu_2) = \sum_{n\neq 0} e^{-bE_n} e^{i\theta S_{3n}} \frac{\langle 0|\hat{\mathcal{W}}_0(\nu_1)|n_{\frac{\theta}{2}}\rangle}{\langle 0|\hat{\mathcal{W}}_0(\nu_1)|0\rangle} \frac{\langle n_{-\frac{\theta}{2}}|\hat{\mathcal{W}}_0(\nu_2)|0\rangle}{\langle 0|\hat{\mathcal{W}}_0(\nu_2)|0\rangle}$$





Sketch of Derivation 6

Take $T \to \infty$

$$\tilde{\mathcal{C}}_E(\theta; b; \nu_1, \nu_2) = \sum_{n \neq 0} e^{-bE_n} e^{i\theta S_{3n}} \frac{\langle 0|\hat{\mathcal{W}}_0(\nu_1)|n_{\frac{\theta}{2}}\rangle}{\langle 0|\hat{\mathcal{W}}_0(\nu_1)|0\rangle} \frac{\langle n_{-\frac{\theta}{2}}|\hat{\mathcal{W}}_0(\nu_2)|0\rangle}{\langle 0|\hat{\mathcal{W}}_0(\nu_2)|0\rangle}$$

$$= \sum_{\alpha \neq 0} \mathcal{P}_\alpha \sum_{\{s_3\}_\alpha} e^{i\theta S_{3\alpha}} \int d\Omega_\alpha \, e^{-bE_\alpha} W_\alpha^+(\{\vec{p}_{\frac{\theta}{2}}\}, \{s_3\}; \nu_1) W_\alpha^-(\{\vec{p}_{-\frac{\theta}{2}}\}, \{s_3\}; \nu_2)$$

E_α, $S_{3\alpha}$: total energy and 3rd component of spin in state α

$\{\vec{p}_{\pm\frac{\theta}{2}}\}$: all momenta rotated around 3-axis

Selection rule: $W_\alpha^\pm$ nonzero only for vanishing discrete charges (electric charge, baryon number, strangeness,...)

$$Q = B = S = \ldots = 0$$

Sketch of Derivation 7

Change of variables ($\sim$ momentum components along the original loops)

$$x_\pm = (\vec{p}_{\pm\frac{\theta}{2}})_1 = \cos\frac{\theta}{2}p_1 \pm \sin\frac{\theta}{2}p_2 \qquad \vec{p}_{\pm\frac{\theta}{2}} = \left(x_\pm, \pm\cot\theta x_\pm \mp \frac{1}{\sin\theta}x_\mp, p_3\right)$$

$$\begin{aligned}\tilde{\mathcal{C}}_E(\theta; b; \nu_1, \nu_2) = \sum_{\alpha\neq 0}\mathcal{P}_\alpha \sum_{\{s_3\}_\alpha} \frac{e^{i\theta S_{3\alpha}}}{(\sin\theta)^{\mathcal{N}_\alpha}} \int d\Omega'_\alpha e^{-bE_\alpha} \\ \times W_\alpha^+(\{(x_+, \cot\theta x_+ - \tfrac{1}{\sin\theta}x_-, p_3)\}, \{s_3\}; \nu_1) \\ \times W_\alpha^-(\{(x_-, -\cot\theta x_- + \tfrac{1}{\sin\theta}x_+, p_3)\}, \{s_3\}; \nu_2)\end{aligned}$$

$$d\Omega_\alpha = \prod_{a,i}\left[\frac{d^3p}{(2\pi)^3 2\varepsilon}\right]_{a,i} \to \prod_{a,i}\frac{1}{\sin\theta}\left[\frac{dx_+ dx_- dp_3}{(2\pi)^3 2\varepsilon}\right]_{a,i} = \frac{d\Omega'_\alpha}{(\sin\theta)^{\mathcal{N}_\alpha}}$$

$$\varepsilon = \sqrt{m^2 + \vec{p}^{\,2}} = \sqrt{m^2 + \left(\frac{x_+ + x_-}{2\cos(\theta/2)}\right)^2 + \left(\frac{x_+ - x_-}{2\sin(\theta/2)}\right)^2 + p_3^2} = \varepsilon(\theta)$$

Analytic Continuation and Large-χ Limit

Assumptions:

1. the analytic continuation can be performed term by term [crucial assumption, requires good convergence properties]
2. $W_\alpha^\pm$ expressed in terms of $x_\pm$ are analytic in θ [$\delta(\sum x_+)$ and $\delta(\sum x_-)$ in $W_\alpha^\pm$ due to translation invariance]

Complex $z = \theta - i\chi$

$$\frac{e^{izS_{3\alpha}}}{(\sin z)^{\mathcal{N}_\alpha}} \int d\Omega'_\alpha e^{-bE_\alpha} W_\alpha^+(\{(x_+, \cot z x_+ - \tfrac{1}{\sin z} x_-, p_3)\}, \{s_3\}; \nu_1) \times W_\alpha^-(\{(x_-, -\cot z x_- + \tfrac{1}{\sin z} x_+, p_3)\}, \{s_3\}; \nu_2)$$

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Integration is convergent as $\mathrm{Re}\ \varepsilon > 0$ for $\theta \in (0, \pi)$ [$E_\alpha = \sum \varepsilon$]

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Take $\theta \to 0$

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Complex $z = \theta - i\chi$

$$\frac{i^{\mathcal{N}_\alpha} e^{\chi S_{3\alpha}}}{(\sinh\chi)^{\mathcal{N}_\alpha}} \int d\Omega'_\alpha e^{-bE_\alpha} W_\alpha^+(\{(x_+, i\coth\chi x_+ - i\tfrac{1}{\sinh\chi}x_-, p_3)\}, \{s_3\}; \nu_1) \times W_\alpha^-(\{(x_-, -i\coth\chi x_- + i\tfrac{1}{\sinh\chi}x_+, p_3)\}, \{s_3\}; \nu_2)$$

Take $\theta \to 0$, $\chi \to \infty$

Analytic Continuation and Large-χ Limit

Assumptions:

1. the analytic continuation can be performed term by term [crucial assumption, requires good convergence properties]
2. $W_\alpha^\pm$ expressed in terms of $x_\pm$ are analytic in θ [$\delta(\sum x_+)$ and $\delta(\sum x_-)$ in $W_\alpha^\pm$ due to translation invariance]

Complex $z = \theta - i\chi$

$$(2i)^{\mathcal{N}_\alpha} e^{\chi(S_{3\alpha}-\mathcal{N}_\alpha)} \int d\Omega'_\alpha e^{-b\tilde{E}_\alpha} W_\alpha^+(\{(x_+, ix_+, p_3)\}, \{s_3\}; \nu_1) \times W_\alpha^-(\{(x_-, -ix_-, p_3)\}, \{s_3\}; \nu_2)$$

Take $\theta \to 0$, $\chi \to \infty$: integrand independent of χ

For $\chi \to \infty$, $\varepsilon \to \sqrt{m^2 + p_3^2} = \tilde{\varepsilon}$, independent of χ; set $\tilde{E}_\alpha = \sum \tilde{\varepsilon}$

Analytic Continuation and Large-χ Limit

Assumptions:

1. the analytic continuation can be performed term by term [crucial assumption, requires good convergence properties]
2. $W_\alpha^\pm$ expressed in terms of $x_\pm$ are analytic in θ [$\delta(\sum x_+)$ and $\delta(\sum x_-)$ in $W_\alpha^\pm$ due to translation invariance]

Complex $z = \theta - i\chi$

$$(2i)^{\mathcal{N}_\alpha} e^{\chi(S_{3\alpha}-\mathcal{N}_\alpha)} \int d\tilde{\Omega}_\alpha \, e^{-b\tilde{E}_\alpha} \int dX_\alpha^+ \, W_\alpha^+(\{(x_+, ix_+, p_3)\}, \{s_3\}; \nu_1) \times \int dX_\alpha^- \, W_\alpha^-(\{(x_-, -ix_-, p_3)\}, \{s_3\}; \nu_2)$$

$$d\tilde{\Omega}_\alpha = \prod_{a,i} \left[\frac{dp_3}{(2\pi)2\tilde{\varepsilon}}\right]_{a,i} \qquad dX_\alpha^\pm = \prod_{a,i} \left[\frac{dx_\pm}{(2\pi)}\right]_{a,i}$$

Analytic Continuation and Large-χ Limit

Assumptions:

1. the analytic continuation can be performed term by term [crucial assumption, requires good convergence properties]
2. $W_\alpha^\pm$ expressed in terms of $x_\pm$ are analytic in θ [$\delta(\sum x_+)$ and $\delta(\sum x_-)$ in $W_\alpha^\pm$ due to translation invariance]

Complex $z = \theta - i\chi$

$$(2i)^{\mathcal{N}_\alpha} e^{\chi(S_{3\alpha}^{\max} - \mathcal{N}_\alpha)} \int d\tilde{\Omega}_\alpha \, e^{-b\tilde{E}_\alpha} \int dX_\alpha^+ \, W_\alpha^+(\{(x_+, ix_+, p_3)\}, \{s\}; \nu_1) \times \int dX_\alpha^- \, W_\alpha^-(\{(x_-, -ix_-, p_3)\}, \{s\}; \nu_2)$$

Leading contribution at large χ from maximal $S_{3\alpha}^{\max} = \sum_a n_a(\alpha) s^{(a)}$, with $s_3^{(a)i} = s^{(a)}$ for all particles

Large-b Expansion

We are interested in $b \gg m^{-1}$: set $p_3 = \frac{\tilde{p}_3}{\sqrt{bm}}$

$$\tilde{\varepsilon} = \sqrt{m^2 + p_3^2} = m\left(1 + \frac{1}{2bm}\left(\frac{\tilde{p}_3}{m}\right)^2 + \mathcal{O}\left(\frac{1}{(bm)^2}\right)\right)$$

$$\frac{dp_3}{2\pi\tilde{\varepsilon}} = \frac{d\tilde{p}_3}{4\pi m\sqrt{bm}}\left(1 + \mathcal{O}\left(\frac{1}{bm}\right)\right)$$

$$\int dX_\alpha^\pm \, W_\alpha^\pm(\{(x_\pm, \pm i x_\pm, \tfrac{\tilde{p}_3}{\sqrt{bm}})\}, \{s_3^{(a)}\}; \nu) = \mathcal{F}_\alpha^\pm(\nu) + \dots$$

For large b the integration in $d\tilde{p}_3$ becomes Gaussian

$$\frac{dp_3}{2\pi\tilde{\varepsilon}} e^{-b\tilde{\varepsilon}} \to \frac{d\tilde{p}_3}{4\pi m\sqrt{bm}} e^{-bm - \frac{1}{2}\left(\frac{\tilde{p}_3}{m}\right)^2} = \frac{e^{-bm}}{2\sqrt{2\pi bm}}$$

If $\mathcal{F}_\alpha^\pm(\nu_{1,2})$ vanish $\to$ extra inverse powers of bm and constant factors

Asymptotic Behaviour of the Correlator

At large χ and large b, setting $\tilde{\mathcal{C}}_M(\chi; b; \nu_1, \nu_2) \equiv \tilde{\mathcal{C}}_E(-i\chi; b; \nu_1, \nu_2)$

$$\tilde{\mathcal{C}}_M(\chi; b; \nu_1, \nu_2) \underset{\chi\to\infty,\, b\to\infty}{\sim} \sum_{\alpha\neq 0} \mathcal{P}_\alpha i^{N_\alpha} \mathcal{F}_\alpha^+(\nu_1)\mathcal{F}_\alpha^-(\nu_2) \prod_a {w_a}^{n_a(\alpha)}$$

up to $\mathcal{O}(e^{-\chi})$ and $\mathcal{O}(b^{-1})$

$$w_a(\chi, b) = \frac{1}{\sqrt{2\pi b m^{(a)}}} e^{\chi[s^{(a)}-1]} e^{-b m^{(a)}} = \frac{1}{\sqrt{2\pi b m^{(a)}}} e^{[R_{\text{eff}}^{(a)}(s) - b] m^{(a)}}$$

Reminiscent of exchange of spin-J particle $\to$ contribution $\propto s^{J-1}$

Contribution of state α non-negligible only for

$$b \lesssim R_{\text{eff}}^{[\alpha]} = \frac{\sum_a n_a(\alpha) m^{(a)} R_{\text{eff}}^{(a)}(s)}{\sum_a n_a(\alpha) m^{(a)}} \qquad R_{\text{eff}}^{(a)}(s) \equiv \frac{s^{(a)} - 1}{m^{(a)}} \chi$$

Extracting the High-Energy Behaviour

$\tilde{\mathcal{C}}_M$ enters the expression for $\sigma_{\rm tot}$

$$\sigma_{\rm tot} = 2{\rm Re} \int d^2\vec{b}_\perp \, A(s, \vec{b}_\perp) = -4\pi {\rm Re} \int_0^\infty dbb \, \langle\langle \tilde{\mathcal{C}}_M(\chi; b; \nu_1, \nu_2) \rangle\rangle$$

What is the characteristic b_c?

$$b_c(s) = \max_\alpha R_{\rm eff}^{[\alpha]}(s) = \max_a R_{\rm eff}^{(a)}(s) = \left[\max_a \frac{s^{(a)} - 1}{m^{(a)}} \right] \chi = \frac{\tilde{s} - 1}{\tilde{m}} \chi$$

If higher-spin ($s^{(a)} > 1$) stable states exist

$$\sigma_{\rm tot} \propto b_c(s)^2 \sim \left(\frac{\tilde{s} - 1}{\tilde{m}} \right)^2 \log^2 s$$

If $\tilde{s} < 1$, $\sigma_{\rm tot} \to 0$; if $\tilde{s} = 1$, $\sigma_{\rm tot} \to {\rm const.}$

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Scaling Variable

Can we get something more? Change variables to

$$z = \frac{1}{\sqrt{\chi}} e^{\chi(\tilde{s}-1)} e^{-\tilde{m}b} \qquad \frac{dz}{z} = -\tilde{m}db$$

and take $\chi \to \infty$ with z fixed

$$w_a(\chi, z) = \exp\left\{ m^{(a)}\chi \left[\frac{s^{(a)}-1}{m^{(a)}} - \frac{\tilde{s}-1}{\tilde{m}} \right] \right\} \frac{z^{\frac{m^{(a)}}{\tilde{m}}}}{\sqrt{2\pi \frac{m^{(a)}}{\tilde{m}} \chi^{-\frac{m^{(a)}}{\tilde{m}}} \log \frac{e^{\chi[\tilde{s}-1]}}{\sqrt{\chi} z}}}$$

$$\to \begin{cases} 0 & (m^{(a)}, s^{(a)}) \neq (\tilde{m}, \tilde{s}) \\ \dfrac{z}{\sqrt{2\pi[\tilde{s}-1]}} & (m^{(a)}, s^{(a)}) = (\tilde{m}, \tilde{s}) \end{cases}$$

Only states containing particle/antiparticle $(\tilde{m}, \tilde{s})$ survive

Scaling Function

Large-χ, b behaviour encoded in

$$\lim_{\chi,b\to\infty} \tilde{\mathcal{C}}_M(\chi; b; \nu_1, \nu_2)\big|_{z\,\mathrm{fixed}} = g(z; \nu_1, \nu_2) - 1$$

$\langle\langle \tilde{\mathcal{C}}_M \rangle\rangle$ bounded (unitarity) for all physical processes $\to$ expect $\tilde{\mathcal{C}}_M$ bounded
$\to |g(z; \nu_1, \nu_2)| \leq 1$ bounded $\forall\ z, \nu_1, \nu_2$

$$\sigma_{\mathrm{tot}} \simeq \frac{4\pi}{\tilde{m}^2} \mathrm{Re}\,\langle\langle J \rangle\rangle \qquad J = \int_0^{e^\eta} \frac{dz}{z} \log \frac{e^\eta}{z} \left[1 - g(z; \nu_1, \nu_2)\right]$$

$$\eta = \chi(\tilde{s} - 1) - \tfrac{1}{2} \log \chi \sim (\tilde{s} - 1) \log s$$

Look for $\mathcal{O}(\eta^2)$ terms in J

$$J = \frac{1}{2}\eta^2 [1 - g_\infty(\nu_1, \nu_2)] + \mathcal{O}(\eta)$$

Universal "Froissart-like" Total Cross Section

$$\sigma_{\rm tot} \underset{s\to\infty}{\sim} \frac{2\pi(\tilde{s}-1)^2}{\tilde{m}^2}\left[1-\mathrm{Re}\,\langle\!\langle g_\infty(\nu_1,\nu_2)\rangle\!\rangle\right]\eta^2+\mathcal{O}(\eta)$$

Bound on $\sigma_{\rm tot}$

$$\sigma_{\rm tot} \underset{s\to\infty}{\lesssim} \frac{4\pi(\tilde{s}-1)^2}{\tilde{m}^2}\log^2\frac{s}{m^2}$$

If $\mathrm{Re}\,g_\infty(\nu_1,\nu_2)=0$ ("black disk assumption"), $\sigma_{\rm tot}$ is universal and the leading $\mathcal{O}(\eta^2)$ term is entirely determined by the spectrum

$$\sigma_{\rm tot} \underset{s\to\infty}{\sim} \frac{2\pi(\tilde{s}-1)^2}{\tilde{m}^2}\log^2\frac{s}{m^2}$$

True also if g oscillates at infinity, unaffected by small-b behaviour

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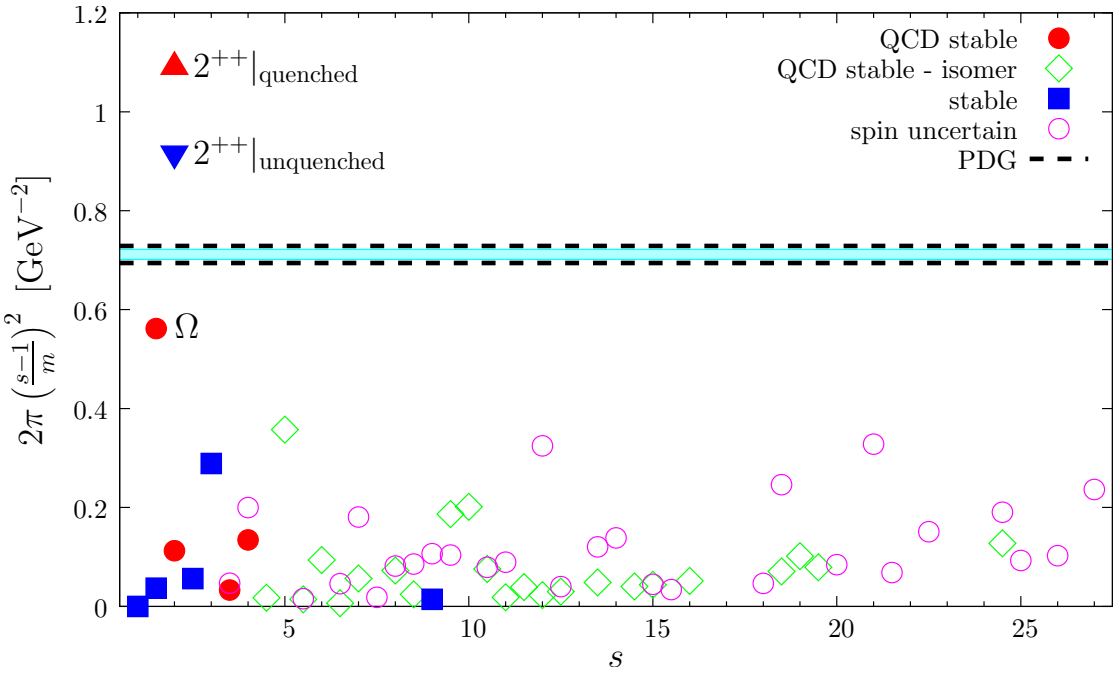
Hadronic Spectrum and Total Cross Sections

Maximise $\frac{s^{(a)}-1}{m^{(a)}}$ over asymptotic stable states of QCD *in isolation*



Data from [Nubase (2003), Gregory *at al.* (2012)]

$\Omega^{\pm}$ baryon, $m_{\Omega^{\pm}} \simeq 1.67$ GeV, $J^P = \frac{3}{2}^+$, $|Q| = 1$, $|S| = 3$

$B_{\text{exp}} \simeq 0.69 \div 0.73$ GeV^{-2} vs. $B_{\text{th}} \simeq 0.56$ GeV^{-2}

Glueball spectrum: $B_Q \gtrsim 1.6 B_{\text{exp}}$, large unquenching effects?

"Froissart-like" Bound

Froissart-Łukaszuk-Martin bound

$$\lim_{s\to\infty} \frac{\sigma_{\rm tot}}{\log^2 \frac{s}{m^2}} \leq \frac{\pi}{m_\pi^2} \simeq 59 \text{ mb}$$

Our "Froissart-like" bound is much more restrictive

$$\lim_{s\to\infty} \frac{\sigma_{\rm tot}}{\log^2 \frac{s}{m^2}} \leq 2B_{\rm th} = \frac{\pi}{m_\Omega^2} \simeq 0.44 \text{ mb}$$

In the $N_f = 2$ chiral limit our "Froissart-like" bound is stable

- masses of nuclei, baryons, and non-Goldstone mesons are expected to change only by a few MeV
- the presence of massless pions can at most make some particle unstable, not the other way around
- Ω expected to remain stable and with approximately the same mass, so it is expected to still be the dominant particle

[MG and E. Meggiolaro, in preparation]

Conclusions and Outlook

Main results:

- $\sigma_{\rm tot} \sim B \log^2 s$, if higher-spin stable states exist
- "Froissart-like" bound $B \leq \frac{4\pi}{\mu^2}$, $\frac{1}{\mu} = \max \frac{s^{(a)}-1}{m^{(a)}}$ determined by the spectrum with $\mu = 2m_\Omega$
  - more restrictive and stable in the chiral limit
- Under the "black disk assumption", $\sigma_{\rm tot}$ is universal and entirely determined from the spectrum, $B_{\rm th} = \frac{2\pi}{\mu^2} \sim B_{\rm exp}$ within $20 \div 25\%$
  - first subleading term in $\sigma_{\rm tot}$ is $\propto \log s \cdot \log\log s$ and universal
  - universal black-disk amplitude for elastic scattering

Open issues:

- Why large unquenching effects?
- What relation with Regge theory?

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Subleading Terms

$$\sigma_{\rm tot} \underset{s\to\infty}{\sim} \frac{2\pi(\tilde{s}-1)^2}{\tilde{m}^2}\left[1-{\rm Re}\,\langle\!\langle g_\infty(\nu_1,\nu_2)\rangle\!\rangle\right]\eta^2+\mathcal{O}(\eta)$$

$$\eta = \chi(\tilde{s}-1) - \frac{1}{2}\log\chi$$

If extra inverse powers $(bm)^{-\frac{\lambda}{2}}$ are present in $W_\alpha, \overline{W}_\alpha$

$$\eta = \chi(\tilde{s}-1) - \frac{1+\lambda}{2}\log\chi$$

First subleading term in $\sigma_{\rm tot}$ is $\propto \chi\log\chi \sim \log s\cdot\log\log s$

Elastic Scattering Amplitudes

If $g_\infty(\nu_1, \nu_2) = 0$

$$\mathcal{M}(s,t) \underset{s\to\infty}{\sim} 4\pi i\, s \frac{\eta}{q\tilde{m}} J_1\left(\frac{q\eta}{\tilde{m}}\right) = i\, s\, \sigma_{\rm tot} \frac{2J_1(\varrho)}{\varrho}$$

$$\varrho = \frac{\sqrt{-t}\eta}{\tilde{m}} \simeq \frac{\sqrt{-t}\log\frac{s}{m_1 m_2}}{\tilde{m}}$$

- Black disk amplitude (purely imaginary):
  - total elastic cross section $\frac{\sigma_{\rm el}}{\sigma_{\rm tot}} = \frac{1}{2}$ $\left[\frac{\sigma_{\rm el}}{\sigma_{\rm tot}}\Big|_{\rm exp,\,\sqrt{s}=7\,\rm TeV} \simeq 0.26\right]$
  - B-slope $\mathcal{B} \equiv \frac{d}{dt}\log\frac{d\sigma_{\rm el}}{dt}\Big|_{t=0}$ satisfies $\frac{8\pi\mathcal{B}}{\sigma_{\rm tot}} = 1$ $\left[\frac{8\pi\mathcal{B}}{\sigma_{\rm tot}}\Big|_{\rm exp,\sqrt{s}=7\,\rm TeV} \simeq 1.97\right]$
  - zero at t_0, $\frac{|t_0|\sigma_{\rm tot}}{2\pi x_0^2} = 1$, $x_0 \simeq 3.8 \approx$ dip in $\frac{d\sigma_{\rm el}}{dt}$ $\left[\frac{|t_{\rm dip}|\sigma_{\rm tot}}{2\pi x_0^2}\Big|_{\rm exp}\right.$ well above 1]

[Csörgő, Nemes (2014)]

- $\mathcal{M}(s,t)/\mathcal{M}(s,0) = 2J_1(\varrho)/\varrho$ depends on t only through ϱ
  - universal function of ϱ
  - entire of order 1 in ϱ

[Auberson, Kinoshita, Martin (1971)]

Resummation and Analytic Continuation

Main assumption: possibility to interchange the order of $\sum_\alpha$ and $\theta \to -i\chi$

Partially justified by the short-range (a) nature of strong interactions

LSZ framework: W_α, $\overline{W}_\alpha \leftrightarrow \int [d^4x e^{ip\cdot x}]^n \langle 0| T\{\hat{\mathcal{W}}\Phi_1(x_1)\dots\Phi_n(x_n)\}|0\rangle$

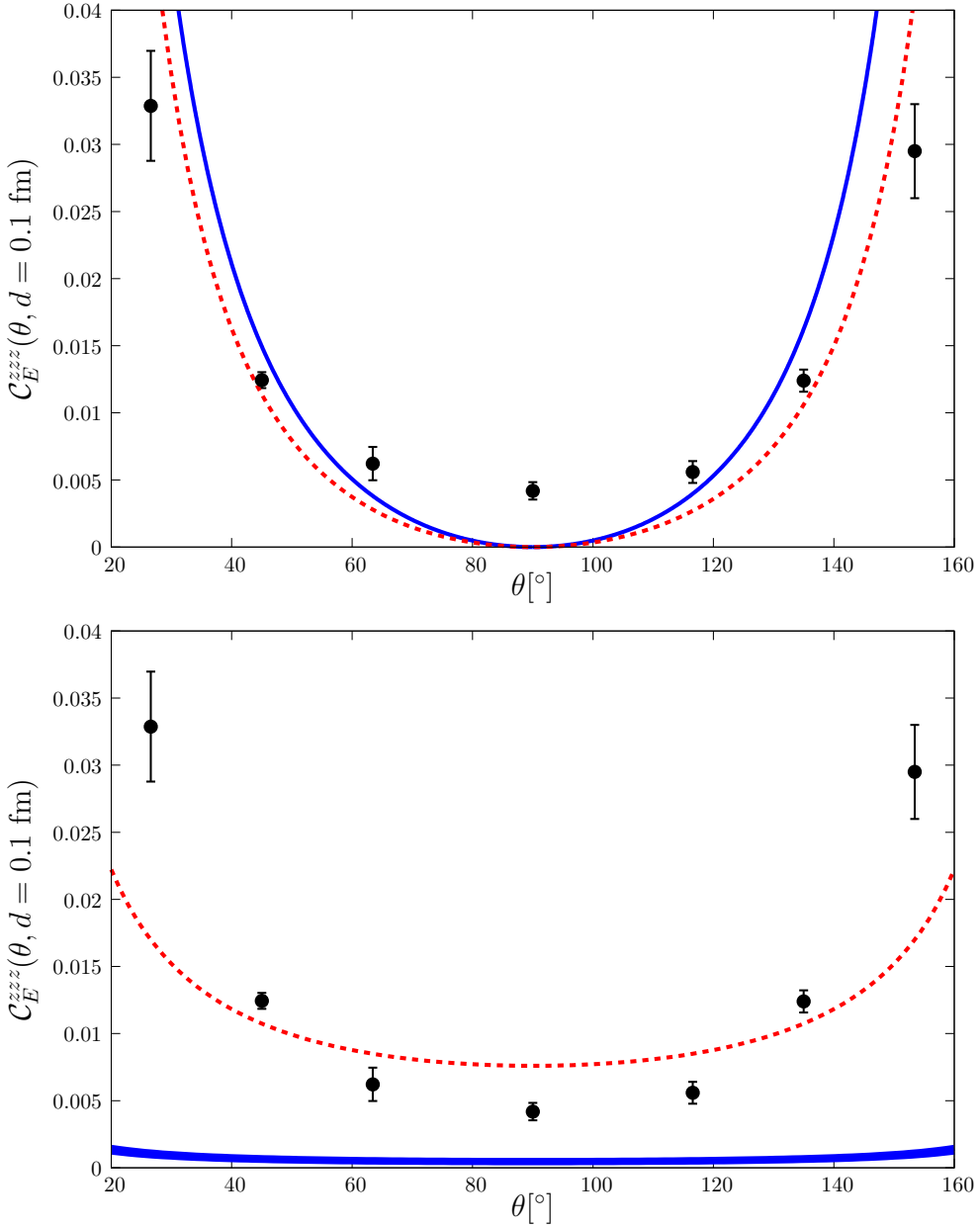
For most of the configurations $|x_i - x_j|/a \gg 1$, mutual interactions negligible, interaction with Wilson loop only local

$$W_\alpha(\{\vec{p}\},\{s_3\};\nu_1) \simeq \prod_{a,i} \frac{\langle 0|\hat{\mathcal{W}}_0(\nu_1)|\vec{p}^{\,(a)i}, s_3^{(a)i}\rangle}{\langle 0|\hat{\mathcal{W}}_0(\nu_1)|0\rangle} = \prod_{a,i} W_a(\vec{p}^{\,(a)i}, s_3^{(a)i};\nu_1)$$

$$\tilde{C}_E \simeq \exp\left\{\sum_a \sum_{s_3} e^{i\theta s_3} \int d\Omega_a\, e^{-b\varepsilon^{(a)}} W_a(\vec{p}_{a\frac{\theta}{2}}, s_3;\nu_1)\overline{W}_a(\vec{p}_{a-\frac{\theta}{2}}, s_3;\nu_2)\right\} - 1$$

Resummation and analytic continuation commute

NP Models and Lattice Results



Lattice calculations give "true" prediction of QCD (within errors) $\Rightarrow$ test analytic NP calculations

Numerical predictions and fits of model functions of SVM (top) and ILM (bottom) to lattice data

Lattice setup:

- Wilson action for $SU(3)$ gauge theory (*quenched* QCD)
- 16^4 hypercubic lattice, $a \simeq 0.1\,\mathrm{fm}$
- longest available loops ($L \simeq 8$)
- $\cot\theta = 0, \pm 1/2, \pm 1, \pm 2$
- $|\vec{r}_{1,2\perp}| = 1a$, $|\vec{d}_\perp| = 0, 1, 2a$
- "zzz": $\vec{d}_\perp \parallel \vec{r}_{1\perp} \parallel \vec{r}_{2\perp}$
- "zyy": $\vec{d}_\perp \perp \vec{r}_{1\perp} \parallel \vec{r}_{2\perp}$
- "ave": average over orientations

Rising Cross Sections From the Lattice

Parameterisation: $\mathcal{C}_E = e^{K_E} - 1$

$$K_E = \sum_i f_i(\theta) g_i(\vec{b}_\perp, \nu_1, \nu_2)$$

Unitarity constraint: $\mathrm{Re}\ K_M \leq 0$

$$[K_M(\chi) = K_E(\theta \to -i\chi)]$$

At large b, $K_E, K_M \sim (\sum) e^{-\mu b}$

If $K_M \sim \chi^p e^{n\chi}\, e^{-\mu b} \sim (\log s)^p s^n\, e^{-\mu b}$

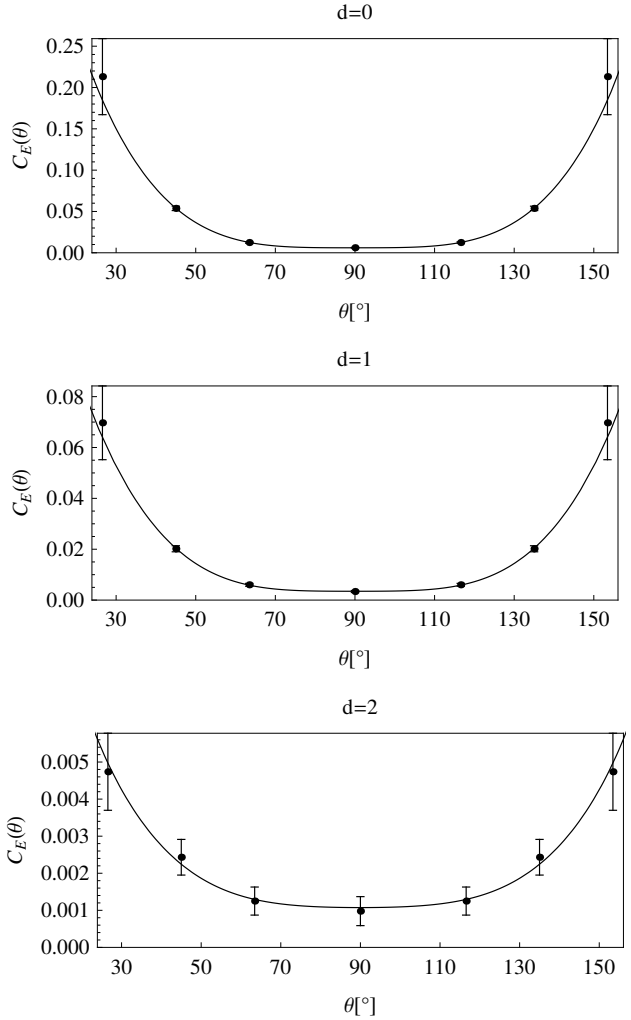
$$\sigma_{\text{tot}}^{(hh)} \sim B \log^2 s$$

with $\quad B = \frac{2\pi n^2}{\mu^2} \quad$ **universal**

Estimate of B fairly agrees with B_{exp}

(although *quenched* and with rather large errors)

Where does this come from?



$$K_E = \frac{K_1}{\sin\theta} + K_2\left(\tfrac{\pi}{2} - \theta\right)^3 \cos\theta$$

More on the Scaling Function

Two possible kinds of dominant particle

(1) Self-conjugate boson ($Q = B = S = \ldots = 0$)

(2) Boson with nonzero charges, or fermion

$$\tilde{\mathcal{C}}_M \to g(z;\nu_1,\nu_2) - 1 = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n!} C_n^0(\nu_1,\nu_2) z^n & (1) \\ \displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n)!} C_n(\nu_1,\nu_2) z^{2n} & (2) \end{cases}$$

Universal "Froissart-like" Total Cross Section

$$\sigma_{\rm tot} \simeq \frac{4\pi}{\tilde{m}^2}{\rm Re}\,\langle\langle J\rangle\rangle \qquad J = \int_0^{e^\eta} \frac{dz}{z}\,\log\frac{e^\eta}{z}\,[1 - g(z;\nu_1,\nu_2)]$$

$$e^\eta = \frac{e^{\chi(\tilde{s}-1)}}{\sqrt{\chi}} \qquad \eta = \chi(\tilde{s}-1) - \tfrac{1}{2}\log\chi$$

Look for $\mathcal{O}(\eta^2)$ terms in $J = J_1 - J_2 + J_3$

$$J_1 = \int_1^{e^\eta} \frac{dz}{z}\,\log\frac{e^\eta}{z} = \frac{1}{2}\eta^2$$

$$J_2 = \int_1^{e^\eta} \frac{dz}{z}\,\log\frac{e^\eta}{z}\,g(z;\nu_1,\nu_2) = \frac{1}{2}\eta^2 g_\infty(\nu_1,\nu_2) + \mathcal{O}(\eta)$$

$$J_3 = \int_0^{1} \frac{dz}{z}\,\log\frac{e^\eta}{z}\,[1 - g(z;\nu_1,\nu_2)] = \mathcal{O}(\eta)$$