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Size scaling of failure strength with fat-tailed disorder in a fiber bundle model

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We investigate the size scaling of the macroscopic fracture strength of heterogeneous materials when microscopic disorder is controlled by fat-tailed distributions. We consider a fiber bundle model where the strength of single fibers is described by a power law distribution over a finite range. Tuning the amount of disorder by varying the power law exponent and the upper cutoff of fibers' strength, in the limit of equal load sharing an astonishing size effect is revealed: For small system sizes the bundle strength increases with the number of fibers, and the usual decreasing size effect of heterogeneous materials is restored only beyond a characteristic size. We show analytically that the extreme order statistics of fibers' strength is responsible for this peculiar behavior. Analyzing the results of computer simulations we deduce a scaling form which describes the dependence of the macroscopic strength of fiber bundles on the parameters of microscopic disorder over the entire range of system sizes.

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I. INTRODUCTION

The disorder of materials plays a crucial role in their fracture processes under mechanical loading. Strength fluctuations of local material elements can lead to crack nucleation at low loads reducing the failure strength compared to homogeneous materials [1–3]. Additionally, disorder gives rise to sample-to-sample fluctuations of fracture strength with an average value which depends on the system size [3]. This so-called size effect of the fracture strength of materials has great importance for applications: on the one hand it has to be taken into account in engineering design of large-scale construction, and on the other hand, it controls how results of laboratory measurements can be scaled up to real constructions and to the scale of geological phenomena [1–4].

The statistics of fracture strength and the associated size effect are usually described by extreme value theory [5], which relates the macroscopic strength of materials to the statistics of weakest microscopic regions [3,6,7]. Weibull gave the first quantitative explanation of the statistical size effect formulating the weakest link idea, namely, the volume element of the weakest flaw drives the failure of the entire system, and he determined the probability distribution of failure strength of macroscopic samples. In order to investigate how the enhanced stress around cracks and the interaction between cracks affect the strength, stochastic lattice models of materials have been widely used [2,3,8–11]. In these models disorder is represented either by random dilution of regular lattices or by the random strength of cohesive elements. Such model calculations confirmed that extreme value statistics describes the distribution of macroscopic strength; however, the general validity of the Weibull distribution has been questioned, although it is widely used in engineering design [12].

To study the size scaling of fracture strength the fiber bundle model (FBM) also provides an adequate framework [13–18]. In FBMs the sample is discretized in terms of parallel fibers where controlling the mechanical response, strength, and interaction of fibers various types of mechanical responses can be represented. Additionally, FBMs are simple enough to obtain analytic solutions for the most important quantities of interest. For the redistribution of load after fiber breaking two limiting cases are very useful to study: the equal (ELS) and local (LLS) load sharing: under ELS the excess load after failure events is equally shared by all the intact fibers, and hence, the stress field remains homogeneous over the entire loading process. For LLS the load dropped by the broken fiber is equally shared by the intact elements of its local neighborhood, resulting in a high stress concentration along broken clusters of fibers.

For ELS, analytic calculations have revealed [15,19,20] that in the limit of large bundle size N the average values $\langle\sigma_c\rangle$ and $\langle\varepsilon_c\rangle$ of the fracture stress σ_c and strain ε_c converge to finite values according to a power law functional form:

$$\langle\sigma_c\rangle(N) = \sigma_c(\infty) + AN^{-\alpha}, \tag{1}$$

$$\langle\varepsilon_c\rangle(N) = \varepsilon_c(\infty) + BN^{-\alpha}. \tag{2}$$

Here $\sigma_c(\infty)$ and $\varepsilon_c(\infty)$ denote the asymptotic bundle strength. The scaling exponent α has the value $\alpha = 2/3$, which proved to be universal for a broad class of disorder distributions, while the multiplication factors A and B depend on the specific type of disorder [15,19,20].

For LLS, numerical calculations showed that the macroscopic strength of bundles, where the strength distribution of single fibers expands to zero, diminishes as the system size N increases. The convergence to zero strength is logarithmically slow with the functional form

$$\langle\sigma_c\rangle(N) \sim 1/(\ln N)^{\beta}, \tag{3}$$

where the exponent β was found to depend on the precise range of load sharing [21–28]. For some modalities of stress transfer an even slower asymptotic convergence $\langle\sigma_c\rangle(N) \sim 1/\ln(\ln N)$ to zero strength was found such as for hierarchical load transfer [29]. The effect of the range of load sharing on the fracture strength of fiber bundles has been studied for a moderate amount of disorders where the strength of single fibers is typically sampled from a uniform, exponential, or Weibull distribution. However, the precise amount of disorder

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may have a strong effect on the size scaling of fracture strength, which has not been explored.

In the present paper we investigate the effect of the amount of microscale disorder on the size scaling of the macroscopic strength of equal load-sharing fiber bundles focusing on the limiting case of extremely high disorder. We consider a power law distribution of fibers' strength over a finite range where the amount of disorder can be controlled by the exponent and by the upper cutoff of the strength values. As the most remarkable result, our study revealed that in a range of parameters the bundle strength increases with the system size. The usual decreasing behavior sets in only beyond a characteristic system size which depends on the amount of disorder. We give a quantitative explanation of these findings in terms of extreme order theory. The results may have potential applications for material design.

II. FIBER BUNDLE MODEL WITH FAT-TAILED DISORDER

In our model we consider a bundle of N parallel fibers, which are assumed to have a perfectly brittle behavior, i.e., they exhibit a linearly elastic response with a Young modulus E up to breaking at a threshold load σ_{th}. The Young modulus is assumed to be constant $E = 1$ such that the disorder of the material is solely represented by the randomness of the breaking threshold σ_{th}: to each fiber a threshold value is assigned σ_{th}^i, $i = 1,\dots,N$ sampled from the probability density $p(\sigma_{th})$. The amount of disorder in the system can be controlled by varying the range $\sigma_{th}^{\min} \leqslant \sigma_{th} \leqslant \sigma_{th}^{\max}$ of strength values and by the functional form of $p(\sigma_{th})$.

In order to explore the effect of extremely high disorder, we consider a power law distribution of threshold values over a finite range. The probability density function is written in the form

$$p(\sigma_{th}) = \begin{cases} 0, & \sigma_{th} < \sigma_{th}^{\min}, \\ A\sigma_{th}^{-(1+\mu)}, & \sigma_{th}^{\min} \leqslant \sigma_{th} \leqslant \sigma_{th}^{\max}, \\ 0, & \sigma_{th}^{\max} < \sigma_{th}, \end{cases} \tag{4}$$

where the lower bound of thresholds $\sigma_{th}^{\min}$ was fixed to $\sigma_{th}^{\min} = 1$. The amount of disorder is controlled by varying the exponent μ of the power law and the upper bound $\sigma_{th}^{\max}$ of the breaking thresholds, while all other parameters are fixed. The value of the exponent is varied over the interval $0 \leqslant \mu \leqslant 1$ because in the limiting case of an infinite upper bound $\sigma_{th}^{\max} \to \infty$ at these μ values the disorder is so high that the thresholds do not have a finite average. For finite values of $\sigma_{th}^{\max}$, of course, the average $\langle\sigma_{th}\rangle$ is always finite; however, the specific values of $\sigma_{th}^{\max}$ and μ have a very strong effect on the behavior of the system both on the macro- and microscales.

After normalizing the probability density $p(\sigma_{th})$ the cumulative distribution function $P(\sigma_{th})$ can be cast into the form

$$P(\sigma_{th}) = \begin{cases} 0 & \sigma_{th} < \sigma_{th}^{\min}, \\ \dfrac{\sigma_{th}^{-\mu} - \left(\sigma_{th}^{\min}\right)^{-\mu}}{\left(\sigma_{th}^{\max}\right)^{-\mu} - \left(\sigma_{th}^{\min}\right)^{-\mu}}, & \sigma_{th}^{\min} \leqslant \sigma_{th} \leqslant \sigma_{th}^{\max}, \\ 1 & \sigma_{th}^{\max} < \sigma_{th}. \end{cases} \tag{5}$$

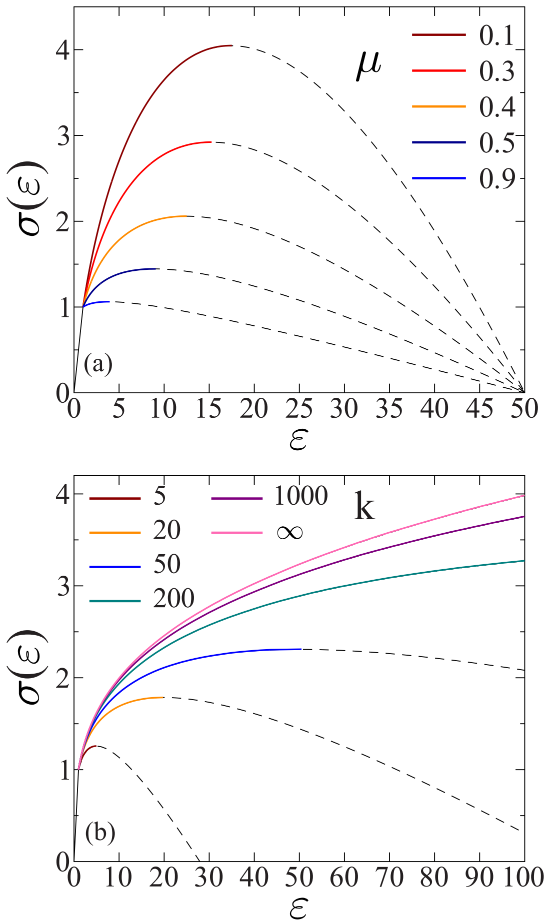

FIG. 1. (a) The macroscopic response $\sigma(\varepsilon)$ of the system for the same upper cutoff $\varepsilon_{\max} = 50$ varying the value of the exponent μ (μ increases from top to bottom). (b) Constitutive curves for a fixed $\mu = 0.7$ exponent varying the upper cutoff $\varepsilon_{\max}$ with the multiplication factor k (k increases from bottom to top). Approaching the phase boundary in both cases the system becomes more and more brittle. The curve on the top of (b) corresponds to the case $\varepsilon_{\max} \to \infty$. The dashed lines represent the full analytical curves of Eq. (6) while the colored dots show the results of stress-controlled simulations.

The macroscopic response of the bundle is characterized by the constitutive equation $\sigma(\varepsilon)$. Assuming equal load sharing, $\sigma(\varepsilon)$ can be cast in the general form $\sigma(\varepsilon) = E\varepsilon[1 - P(E\varepsilon)]$, where the term $1 - P(E\varepsilon)$ provides the fraction of intact fibers at strain ε, which all keep the same load $E\varepsilon$ [15,17,18]. Substituting the cumulative distribution function $P(x)$ from Eq. (5) we arrive at

$$\sigma(\varepsilon) = \begin{cases} \varepsilon, & 0 \leqslant \varepsilon \leqslant \varepsilon_{\min}, \\ \dfrac{\varepsilon\left(\varepsilon^{-\mu} - \varepsilon_{\max}^{-\mu}\right)}{\varepsilon_{\min}^{-\mu} - \varepsilon_{\max}^{-\mu}}, & \varepsilon_{\min} \leqslant \varepsilon \leqslant \varepsilon_{\max}, \\ 0, & \varepsilon_{\max} < \varepsilon, \end{cases} \tag{6}$$

where for clarity the notation $\varepsilon_{\min} = \sigma_{th}^{\min}/E$, $\varepsilon_{\max} = \sigma_{th}^{\max}/E$ was introduced with $E = 1$. The macroscopic constitutive response of the system is illustrated in Fig. 1. Up to the lower bound $\varepsilon_{\min}$ a perfectly linearly elastic response is obtained since no breaking can occur. When the fibers start to break above $\varepsilon_{\min}$, the constitutive curve $\sigma(\varepsilon)$ becomes nonlinear, and

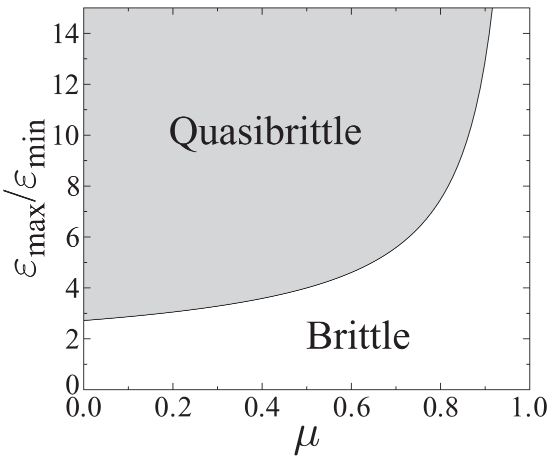

FIG. 2. Phase diagram of the system. The phase boundary separating the brittle and quasibrittle macroscopic response is given by Eq. (9). Note that for $\mu \geqslant 1$ the bundle is always in the brittle phase.

beyond the maximum it decreases to zero as all fibers break gradually.

The fracture strength of the bundle is defined by the value σ_c of the maximum of the constitutive curve and by its position ε_c, called critical stress and strain, respectively. Under stress-controlled loading exceeding the value of σ_c, the bundle rapidly undergoes global failure so that the entire $\sigma(\varepsilon)$ curve can be realized only under strain-controlled loading. Of course, the critical strain depends on the degree of disorder characterized by μ and ε_{max},

$$\varepsilon_c = \varepsilon_{max}(1-\mu)^{1/\mu}, \tag{7}$$

while the critical stress σ_c depends on the lower cutoff ε_{min} as well:

$$\sigma_c = \frac{\mu(1-\mu)^{1/\mu-1}\varepsilon_{max}^{1-\mu}}{\varepsilon_{min}^{-\mu} - \varepsilon_{max}^{-\mu}}. \tag{8}$$

It is a very interesting feature of the system that if the threshold distribution is too narrow, the first fiber breaking already can trigger a catastrophic avalanche of fiber breaking, giving rise to global failure. This occurs when the position of the maximum of the constitutive curve ε_c coincides with the lower bound ε_{min}. Keeping ε_{min} fixed, a threshold value ε_c^{max} of the upper bound ε_{max} can be derived as

$$\varepsilon_{max}^c = \frac{\varepsilon_{min}}{(1-\mu)^{1/\mu}}. \tag{9}$$

It follows that those bundles where $\varepsilon_{max} < \varepsilon_{max}^c$ holds behave in a completely brittle way, i.e., macroscopic failure occurs right after the linear regime of $\sigma(\varepsilon)$ at the instant of the first fiber breaking. However, in the parameter regime $\varepsilon_{max} > \varepsilon_{max}^c$ a quasibrittle response is obtained where macroscopic failure is preceded by breaking avalanches.

It can be observed that as the exponent μ approaches 1 from below, the value of ε_{max}^c diverges so that the regime $\mu \geqslant 1$ is always brittle. The phase diagram of the system is illustrated in Fig. 2.

In the following we analyze how the amount of disorder affects the macroscopic strength of finite bundles in the quasibrittle phase. For clarity, in these calculations the upper cutoff ε_{max} will be expressed in terms of ε_{max}^c as $\varepsilon_{max} = k\varepsilon_{max}^c$, where the multiplication factor k can take any value in the range $k \geqslant 1$.

III. FRACTURE STRENGTH OF FINITE BUNDLES

The fracture strength characterized by the critical strain ε_c and stress σ_c have been obtained analytically in Eqs. (7) and (8) as function of the parameters of the model μ,ε_{min}, and ε_{max}. These analytical calculations assume an infinite system size so that ε_c and σ_c are the $N \to \infty$ asymptotic strength of the bundle. In order to reveal how the finite size of the bundle N affects the average value of the critical strain $\langle\varepsilon_c\rangle$ and stress $\langle\sigma_c\rangle$ we performed computer simulations varying the number of fibers N over six orders of magnitude. Stress-controlled loading of the bundles was performed until the catastrophic avalanche gave rise to global failure. The critical values ε_c and σ_c were determined as the strain and stress of the last stable configuration of the system (see also Fig. 1).

It can be seen in Fig. 3(a) that for low values of the upper cutoff ε_{max} of the strength of single fibers the average bundle strength $\langle\varepsilon_c\rangle$ monotonically decreases with increasing system size as is expected. However, above a certain value of ε_{max} the macroscopic strength has an astonishing unceasing regime for small system sizes so that the usual decreasing behavior of strength is restored only above a characteristic system size N_c. The horizontal lines in the figure show that in the limit of large N the average strength $\langle\varepsilon_c\rangle$ converges to the analytic asymptotic value of Eq. (7). Note that the characteristic system size N_c, which separates the increasing and decreasing regimes of macroscopic strength, is an increasing function of ε_{max}. The same qualitative behavior is observed for the critical stress $\langle\sigma_c\rangle$ in the inset of Fig. 4, which clearly demonstrates that the fracture strength of the fiber bundle increases for small system sizes when the amount of disorder is sufficiently high. The position of the maximum of $\langle\sigma_c\rangle$ coincides with that of $\langle\varepsilon_c\rangle$.

Figure 3(b) and Fig. 4 demonstrate that rescaling the two axis of Fig. 3(a) and of the inset of Fig. 4 the curves of different cutoff values can be collapsed on a master curve. On the horizontal axis the number of fibers is rescaled with ε_{max}^μ, while along the vertical axis the rescaling is performed with the corresponding asymptotic strength $\varepsilon_c(\infty)$ and $\sigma_c(\infty)$ in the two figures. The good quality collapse implies the scaling structures

$$\langle\varepsilon_c\rangle(N,\varepsilon_{max}) = \varepsilon_c(\infty)\Phi\big(N/\varepsilon_{max}^\mu\big), \tag{10}$$

$$\langle\sigma_c\rangle(N,\varepsilon_{max}) = \sigma_c(\infty)\Psi\big(N/\varepsilon_{max}^\mu\big), \tag{11}$$

where $\Phi(x)$ and $\Psi(x)$ denote the scaling functions. The structure of the scaling functions $\Phi(x)$ and $\Psi(x)$ has the consequence that the characteristic system size N_c depends on the parameters as

$$N_c \sim \varepsilon_{max}^\mu. \tag{12}$$

In Figure 3(b) unity is subtracted from the scaling function $\Phi(x)$, which results in an asymptotic power law decrease. This behavior implies the validity of the functional form

$$\Phi(x) \approx 1 + Cx^{-\alpha} \tag{13}$$

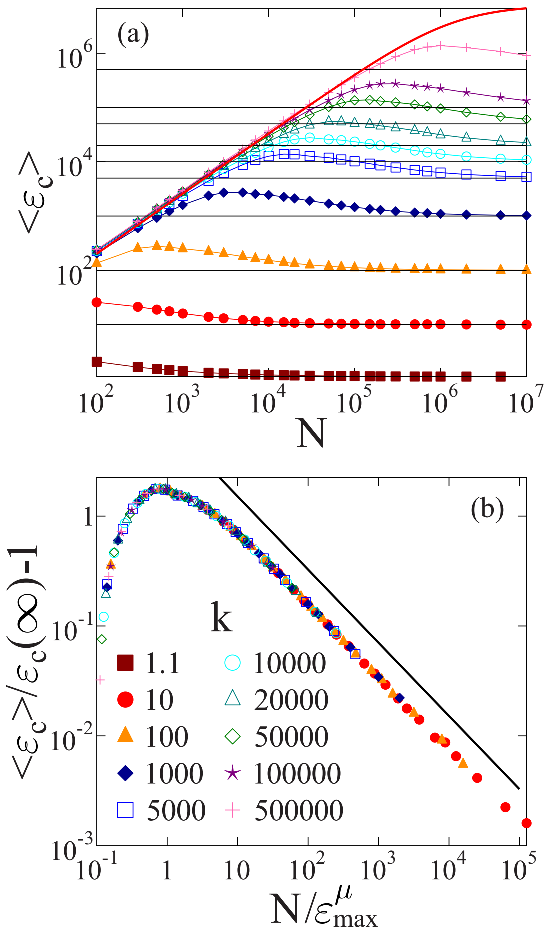

FIG. 3. (a) The average value of the critical strain $\langle\varepsilon_c\rangle$ as a function of the bundle size N for several values of the upper cutoff $\varepsilon_{\max}$ of the strength of single fibers. The horizontal lines represent the corresponding asymptotic strength obtained from Eq. (7). The value of the exponent μ is fixed to $\mu = 0.8$. The upper cutoff $\varepsilon_{\max}$ is parametrized by k such that the legend is the same as in panel (b). The red bold line gives the analytic curve of Eq. (15). (b) Scaling plot of the data presented in panel (a). After rescaling with the asymptotic strength $\varepsilon_c(\infty)$ we subtracted 1 from the result in order to demonstrate the asymptotic power law behavior. The straight line represents a power law of exponent $-2/3$.

for $x > 1$. The value of the exponent was found to be $\alpha = 2/3$, which is consistent with the generic behavior Eq. (1) of the strength of ELS bundles. Note that the scaling function $\Psi(x)$ of $\langle\sigma_c\rangle$ has the same features in Fig. 4 as $\Phi(x)$ so that $\Psi(x)$ can also be described by Eq. (13).

IV. EXTREME ORDER STATISTICS

The peculiar size scaling of macroscopic strength obtained in our simulations is the direct consequence of the fat-tailed strength distribution of single fibers. The main effect of the fat tail is that even for a small system size N the probability to have strong fibers in the bundle can be relatively high. Under equal load-sharing conditions all fibers keep the same load so that fibers break in the increasing order of their breaking thresholds. Our assumption is that for those system sizes N which are along the increasing regime of $\langle\varepsilon_c\rangle(N)$ and $\langle\sigma_c\rangle(N)$ the strongest fibers are so strong that a few of them or even a single one is able to keep the entire load that has been put on the bundle. It follows that the average macroscopic strength $\langle\varepsilon_c\rangle(N)$ should be determined by the average strength of the strongest fiber $\langle\varepsilon_{th}^{\max}\rangle_N$. The average of the largest value of a set of N random numbers sampled from the same probability distribution can be obtained analytically as

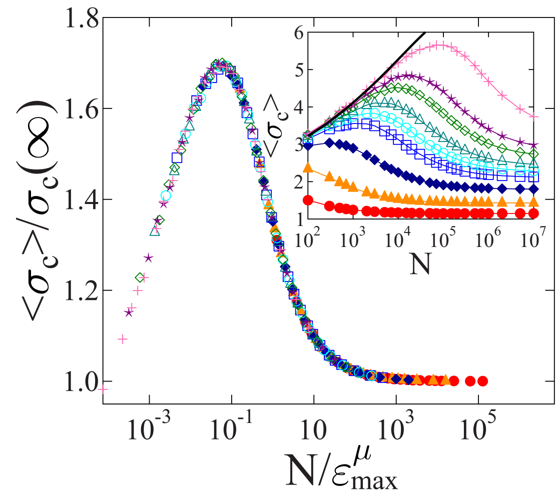

FIG. 4. Inset: The average fracture stress $\langle\sigma_c\rangle(N)$ as a function of the number of fibers N for the same values of the upper cutoff as in Fig. 3 using also the same legend. The bold line represents the curve of Eq. (17). Main panel: Rescaling the two axis of the inset the curves obtained at different cutoff values can be collapsed on a master curve.

$$\langle\varepsilon_c\rangle(N) = \langle\varepsilon_{th}^{\max}\rangle_N = P^{-1}\left(1 - \frac{1}{N+1}\right), \tag{14}$$

where P denotes the cumulative distribution of failure thresholds. Substituting P from Eq. (5), the above expression yields for the macroscopic strength

$$\langle\varepsilon_{th}^{\max}\rangle_N = \left\{\left[\left(\varepsilon_{th}^{\max}\right)^{-\mu} - \left(\varepsilon_{th}^{\min}\right)^{-\mu}\right]\left(1 - \frac{1}{N+1}\right) + \left(\varepsilon_{th}^{\min}\right)^{-\mu}\right\}^{-1/\mu}. \tag{15}$$

It can be observed in Fig. 3 that Eq. (15) provides a high-quality description of the increasing macroscopic fracture strength with the system size. Deviations occur only around the characteristic system size N_c where the curve of $\langle\varepsilon_{th}^{\max}\rangle_N$ saturates since the average of the largest cannot exceed the value of the upper cutoff of fibers' strength $\varepsilon_{\max}$. Note that for large upper cutoffs $\varepsilon_{\max} \to \infty$ Eq. (15) predicts a power law increase of the fracture strain with the system size [30]:

$$\langle\varepsilon_c\rangle(N) \sim N^{1/\mu}. \tag{16}$$

For the fracture stress $\langle\sigma_c\rangle(N)$ it follows from the above arguments that along the increasing branch in Fig. 4 the relation holds:

$$\langle\sigma_c\rangle(N) = \frac{E\langle\varepsilon_c\rangle(N)}{N}. \tag{17}$$

This explains the orders of magnitude difference of $\langle\varepsilon_c\rangle(N)$ and $\langle\sigma_c\rangle(N)$ in Figs. 3 and 4 and the slower increase of the fracture stress:

$$\langle\sigma_c\rangle(N) \sim N^{1/\mu-1}. \tag{18}$$

This relation provides a very good description of the data in the inset of Fig. 4. Beyond the characteristic system size N_c both quantities $\langle\varepsilon_c\rangle(N)$ and $\langle\sigma_c\rangle(N)$ are described by the same size scaling exponent $\alpha = 2/3$. The result shows that for small system sizes the macroscopic strength of the bundle is determined by the extreme order statistics of the strength of single fibers, while above a characteristic system size this behavior breaks down and the average collective behavior of fibers of the bundle dominates [15].

V. DISCUSSION

We investigated the effect of fat-tailed microscopic disorder on the macroscopic fracture strength of heterogeneous materials in the framework of a fiber bundle model with equal load sharing. The amount of disorder was controlled by varying the upper cutoff of fibers' strength and the power law exponent of the strength distribution. Analyzing the constitutive response of the system, we determined its phase diagram on the plane of control parameters: for low values of the upper cutoff the bundle behaves in a completely brittle way where the breaking of the weakest fiber triggers the sudden collapse of the bundle. For sufficiently high disorder a quasibrittle response is obtained where macroscopic failure is approached through stable cracking.

We focused on the size scaling of macroscopic strength of the bundle in the quasibrittle phase. Computer simulations revealed an astonishing size effect: for small system sizes the bundle strength increases with the number of fibers such that the usual decreasing behavior sets on only above a characteristic system size. Fat-tailed disorder has the consequence that at small system sizes strong fibers already are included in the bundle with a high probability. It implies that even a single fiber may be able to keep the total load put on the system so that for small system sizes the macroscopic bundle strength is determined by the extreme order statistics of the strength of single fibers. Since the fiber strength is bounded from above, for large enough system sizes the strongest fiber cannot compete with the load kept by the weaker fibers so that the regular decreasing size scaling gets restored. Based on this argument we could give an analytic description of the size scaling of bundles strength in the presence of fat-tailed disorder and determined the crossover system size, as well.

A similar strengthening behavior for small system sizes has been observed in time-dependent fiber bundles under localized load-sharing conditions [31,32]. It was found that for sufficiently small power law exponents of the life consumption function of fibers, a few long-lived fibers dominate, giving rise to an increased lifetime of the entire bundle. Consequently, the bundle lifetime is described by an extreme value distribution [31,32].

Our study is focused on the ELS limit of FBMs where all fibers keep the same load. In fibrous materials stress fluctuations naturally arise due to the localized load sharing after fiber failures. In the strengthening regime, where extreme order statistics dominates the size scaling, no difference of ELS and LLS systems is expected. However, the crossover system size N_c can depend on the precise form of the load redistribution scheme. Simulation studies in this direction are in progress.

Recently, it has been shown that by controlling the microstructure [33] or the microscale disorder [34] of materials, novel types of materials can be tailored with desired properties for specialized applications. The scaling regime where the macroscopic strength increases with the system size may have potential for material design in future applications.

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