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Crackling noise in sub-critical fracture of heterogeneous materials

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Abstract. We present a theoretical study of the sub-critical fracture of heterogeneous materials under a constant external load. A generic fiber bundle model is proposed, which provides a direct connection between the microscopic fracture mechanisms and the macroscopic time evolution of the sub-critical system. In the model, material elements either fail due to immediate breaking or undergo a damage accumulating ageing process. On the macrolevel the model reproduces the empirical Basquin law of rupture life, and it makes it possible to derive a generic scaling form for the deformation histories of different load values. On the microlevel we found that sub-critical fracture is accompanied by crackling noise, i.e. the competition of the two failure modes of fibers gives rise to a complex bursting activity, where slow damage sequences trigger bursts of breaking events. When the load is equally distributed over the fibers, the size of damage sequences and of bursts, as well as the waiting times in between, are characterized by universal power law distributions, where only the cutoffs have material dependence. When stress concentrations arise in the vicinity of failed regions, the power law distributions of noise characteristics prevail but the exponents are different from their equal load sharing counterparts. In the presence of stress concentration the failure process accelerates resulting in a higher value of the waiting time exponent compared to the case of homogeneous stress distribution.

Keywords: avalanches (theory), fracture (theory)

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1. Introduction

It has long been recognized by industry that structural components exposed to a constant or periodic loading can fail after a finite time even if the load amplitude is far below the safety limit. In everyday life the mysterious sudden breakdown of car or kitchen equipment is a similar experience. Since the load amplitude falls below the fracture strength of the material the phenomenon is called sub-critical fracture. Sub-critical fracture has two main types, i.e. when the external load is constant one speaks about creep rupture, while in the case of periodic loading the phenomenon is called fatigue fracture [1]–[3].

The time dependent fracture of disordered media under sub-critical external loads represents an important applied problem, with intriguing theoretical aspects [1]–[4]. Time dependent fracture plays a crucial role in a broad variety of physical, biological, and geological systems, such as the rupture of adhesion clusters of cells in biomaterials under external stimuli [5], the sub-critical crack growth due to thermal activation of crack nucleation [6]–[8], and the emergence of earthquake sequences [8, 9]. One of the most important macroscopic scaling laws of time dependent fracture is the empirical Basquin law of rupture life which states that the lifetime t_f of samples increases as a power law when the external load amplitude σ_0 decreases [10]:

$$t_f \sim \sigma_0^{-\alpha}. \tag{1}$$

The power law behavior is valid for a broad class of heterogeneous materials; however, the measured values of the Basquin exponent α have a large variation, which implies a strong dependence on material properties [1]–[3], [10].

Laboratory experiments revealed that sub-critical fracture under constant or repeated loading is due to a combination of several mechanisms, among which thermal activation of microcrack nucleation, damage growth, relaxation due to viscoelasticity, and healing of microcracks play an essential role [11]–[14]. Theoretical approaches have serious difficulties in capturing all these mechanisms and in relating the microscopic dynamics to the macroscopic time evolution of the system. Recently, stochastic fracture models and the application of statistical physics have provided a novel insight into the fracture of heterogeneous materials under different loading conditions, but several interesting aspects of sub-critical fracture remained unexplored [1]–[4], [12]–[16].

In this paper we present a detailed theoretical study of the sub-critical failure of heterogeneous materials under a constant external load focusing on the microscopic process of fracture and on the properties of crackling noise accompanying failure. To obtain a theoretical understanding of the failure process, we extended the classical fiber bundle model [17]–[24] by introducing time dependent damage accumulation of fibers so that the model captures the stochastic nature of the fracture process, the immediate breaking of material elements and the cumulative effect of the loading history [25, 26]. We demonstrate that the model reproduces the empirical Basquin law of rupture life; furthermore, the deformation histories of loaded samples obey a generic scaling form. As a continuation of our former studies [25, 26], we show that the sub-critical fracture of heterogeneous materials is accompanied by crackling noise: the separation of timescales of the two competing failure mechanisms of fibers leads to a complex bursting activity on the microscale, where slowly proceeding damage sequences trigger bursts of breakings. In the case of equal load sharing, the size of damage sequences and of bursts, as well as the waiting times in between, are characterized by universal power law distributions. In the more realistic situation of stress concentrations around failed fibers, the power law functional forms of the noise characteristics prevail; however, the exponents are different from their equal load sharing counterparts. Stress concentrations make the failure process faster, so large waiting times become less frequent, increasing the value of the waiting time exponent.

2. Damage accumulation and healing in fiber bundles

In order to give a theoretical description of sub-critical fracture of heterogeneous materials, we have recently worked out an extension of the classical fiber bundle model. In the following we summarize the main components of the model construction based on [25, 26]. We consider a bundle of parallel linear elastic fibers with the same Young modulus E. When the bundle is subjected to a constant external load σ_0 the fibers gradually fail due to two physical mechanisms: fiber i ($i = 1, \ldots, N$) breaks instantaneously at time t when its local load $p_i(t)$ exceeds the tensile strength p_{th}^i of the fiber. Those fibers which remained intact (did not break immediately) undergo a damage accumulation process due to the load that they have experienced. The amount of damage Δc_i that occurred under the load $p_i(t)$ in a time interval Δt is assumed to have the form

$$\Delta c_i = a p_i(t)^{\gamma} \Delta t; \tag{2}$$

hence, the total accumulated damage $c_i(t)$ until time t can be obtained by integrating over the entire loading history of fibers. The damage law equation (2) assumes a power law relation of the rate of damage Δc_i and of the local load p_i, which is analogous to the usual relation leading to the Weibull distribution of the materials' strength [27]. Experiments have shown that healing of microcracks can play an important role in the time evolution of certain materials especially at low load levels. Healing can be captured in the model by introducing a finite range τ for the memory, over which the loading history contributes to the accumulated damage [15]. In a broad class of materials the healing of microcracks typically leads to an exponential form of the memory term [2]. Hence, the total amount of damage accumulated up to time t also taking into account the healing of microcracks can be cast into the final form

$$c_i(t) = a \int_0^t \mathrm{e}^{-(t-t')/\tau} p_i^{\gamma}(t')\, \mathrm{d}t'. \tag{3}$$

The exponent $\gamma > 0$ controls the rate of damage accumulation, and $a > 0$ is a scale parameter. In principle, the range of memory τ can take any positive value $\tau > 0$ such that during the time evolution of the bundle the damage accumulated during the time interval $t' < t - \tau$ heals. The fibers can only tolerate a finite amount of damage and break when $c_i(t)$ exceeds a threshold value c_{th}^i. Each fiber is characterized by two breaking thresholds p_{th}^i and c_{th}^i which are random variables with a joint probability density function $h(p_{\mathrm{th}}, c_{\mathrm{th}})$. Assuming independence of the two breaking modes, the joint density function h can be factorized into a product

$$h(p_{\mathrm{th}}, c_{\mathrm{th}}) = f(c_{\mathrm{th}})g(p_{\mathrm{th}}), \tag{4}$$

where $f(c_{\mathrm{th}})$ and $g(p_{\mathrm{th}})$ are the probability densities and $F(c_{\mathrm{th}})$ and $G(p_{\mathrm{th}})$ the cumulative distributions of the breaking thresholds p_{th} and c_{th}, respectively.

After failure events, the load of the broken fibers has to be overtaken by the remaining intact ones. As a first step, we assume that the excess load is equally redistributed over the intact fibers in the bundle irrespective of their distance from the failure point. Under this equal load sharing (ELS) assumption important characteristic quantities of the system can be obtained by analytic means [17]–[24]. Later on we make the treatment more realistic via localized load sharing (LLS) where the excess load is redistributed in the close vicinity of the broken fiber. Under a constant tensile load σ_0, the load on a single fiber p_0 is initially determined by the quasi-static constitutive equation of FBM [17]–[24]:

$$\sigma_0 = [1 - G(p_0)]\, p_0, \tag{5}$$

which means that fibers with breaking thresholds $p_{\mathrm{th}}^i < p_0$ immediately break. It follows that the external load σ_0 must fall below the tensile strength of the bundle $\sigma_0 < \sigma_{\mathrm{c}}$; otherwise the entire bundle will fail immediately at the instant of the application of the load. This feature is valid irrespective of the range of load sharing, so in LLS bundles the corresponding critical load, which is lower than that of the ELS system, has to be considered. As time elapses, the fibers accumulate damage and break due to their finite damage tolerance. These breakings, however, increase the load on the remaining intact fibers which in turn induce again immediate breakings. In this way, in spite of the independence of the threshold values p_{th} and c_{th}, the two breaking modes are dynamically coupled, gradually driving the system to macroscopic failure in a finite time t_{f} at any load values σ_0. Finally, the evolution equation of the system under equal load sharing conditions can be cast in the form

$$\sigma_0 = \left[1 - F\left(a \int_0^t \mathrm{e}^{-(t-t')/\tau} p(t')^{\gamma}\, \mathrm{d}t'\right)\right] [1 - G(p(t))]\, p(t), \tag{6}$$

where the integral in the argument of F provides the accumulated damage at time t taking into account the finite range of memory [15]. Equation (6) is an integral equation which has to be solved for the load $p(t)$ on the intact fibers at a given external load σ_0 with the initial condition $p(t = 0) = p_0$ obtained from equation (5). The product in equation (6)

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arises due to the independence of the two breaking thresholds. With minor simplifying assumptions, equation (6) can also be solved analytically. In order to make the model more realistic, in the present paper we also consider the case of localized load sharing which can only be investigated by computer simulations.

3. Macroscopic time evolution

On the macrolevel the fracture process is characterized by the evolution of deformation $\varepsilon(t)$ of the specimen, which is related to $p(t)$ as $p(t) = E\varepsilon(t)$, where $E = 1$ is the Young modulus of fibers. Neglecting immediate breaking and healing, the equation of motion of the system equation (6) can be transformed into a differential equation for the number N_{b} of broken fibers as

$$\frac{\mathrm{d}N_{\mathrm{b}}}{\mathrm{d}t} = af(c(t))p(t)^{\gamma}N, \tag{7}$$

where

$$p(t) = \frac{N\sigma_0}{(N - N_{\mathrm{b}}(t))}. \tag{8}$$

This equation system has to be solved for $N_{\mathrm{b}}(t)$ with the initial condition $N_{\mathrm{b}}(t = 0) = 0$, from which the deformation $\varepsilon(t)$ can be determined using equation (8). For uniformly distributed threshold values equation (7) becomes independent of the accumulated damage $c(t)$ and the exact solution of the equation of motion can simply be obtained as

$$\varepsilon(t) = \sigma_0 \left[\frac{t_{\mathrm{f}} - t}{t_{\mathrm{f}}}\right]^{-1/(1+\gamma)}, \tag{9}$$

where

$$t_{\mathrm{f}} = \frac{\sigma_0^{-\gamma}}{a(1+\gamma)}. \tag{10}$$

Equation (9) shows that damage accumulation leads to a finite time singularity where the deformation $\varepsilon(t)$ of the system has a power law divergence with an exponent determined by γ. Macroscopic failure occurs at a finite time t_{f} which defines the lifetime of the system equation (10). It is important to emphasize that t_{f} has a power law dependence on the external load σ_0 in agreement with Basquin's law, equation (1), found experimentally for a broad class of materials [1, 2, 10, 25]. The analytic solution equation (10) shows that the Basquin exponent of the model coincides with that of the microscopic damage law $\alpha = \gamma$.

Another interesting outcome of the derivation is that the macroscopic deformation $\varepsilon(t)$ of a specimen undergoing sub-critical fracture obeys the generic scaling form

$$\varepsilon(t) = \sigma_0^{\delta} S(t\sigma_0^{\beta}), \tag{11}$$

where the scaling function S has a power law divergence as a function of the rescaled time-to-failure

$$S(t\sigma_0^{\beta}) \sim (t_a - t\sigma_0^{\beta})^{-1/(1+\gamma)}, \qquad \text{with } t_a = a(1+\gamma), \tag{12}$$

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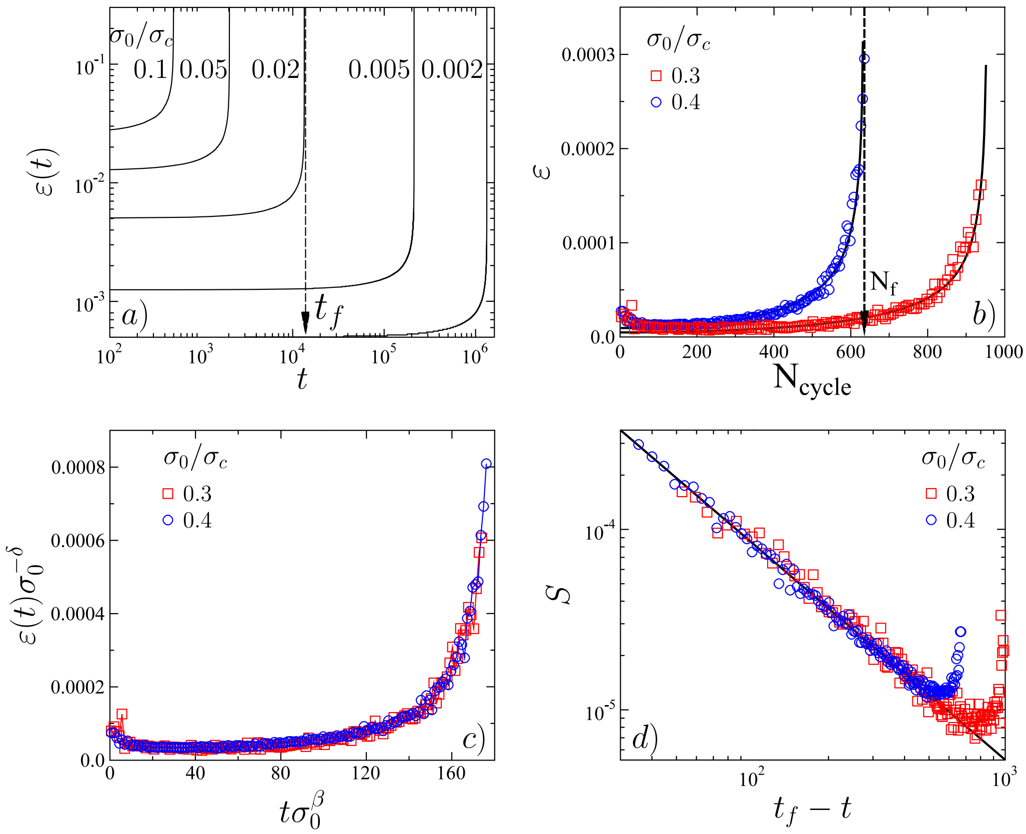

Figure 1. (a) Deformation as a function of time $\varepsilon(t)$ obtained at different load values σ_0 with equation (10). Lowering the external load, the lifetime $t_{\rm f}$ of the system increases. (b) The analytic solution (continuous line) provides an excellent fit of the experimental data (symbols) obtained for asphalt specimens under periodic loading at different load amplitudes [25]. (c) On the basis of the scaling form equation (11), the deformation–time diagrams $\varepsilon(t)$ obtained at different loads can be collapsed on the top of each other. (d) The scaling function S has a power law divergence as a function of time-to-failure $t_{\rm f} - t$.

and the scaling exponents have the values $\delta = 1$ and $\beta = \gamma$. Figure 1(a) presents examples of the solution $\varepsilon(t)$ of equation (6) obtained for breaking thresholds uniformly distributed in the interval $[0, 1]$ (i.e. $g(p_{\rm th}) = 1$ and $f(c_{\rm th}) = 1$) at different ratios $\sigma_0/\sigma_{\rm c}$ setting $\tau \to \infty$ (no healing is considered). The lifetime of one of the samples is indicated by the vertical arrow. In can be seen that the results are in a nice qualitative agreement with the experimental findings, i.e. the deformation is a monotonically increasing function of time with an increasing derivative when the point of macroscopic failure is approached. Lowering the external load σ_0 the lifetime $t_{\rm f}$ of the bundle increases. Since no healing is taken into account, the specimen fails in the simulations under any finite load $\sigma_0 > 0$, just the lifetime $t_{\rm f}$ takes very large values.

In figures 1(b)–(d) the theoretical results are compared to the experimental findings on asphalt specimens of [25]. In the experiments, cylindrical asphalt specimens were subject to diametrical compression applied periodically with a constant amplitude. The failure process was followed by measuring the accumulated deformation at the end of

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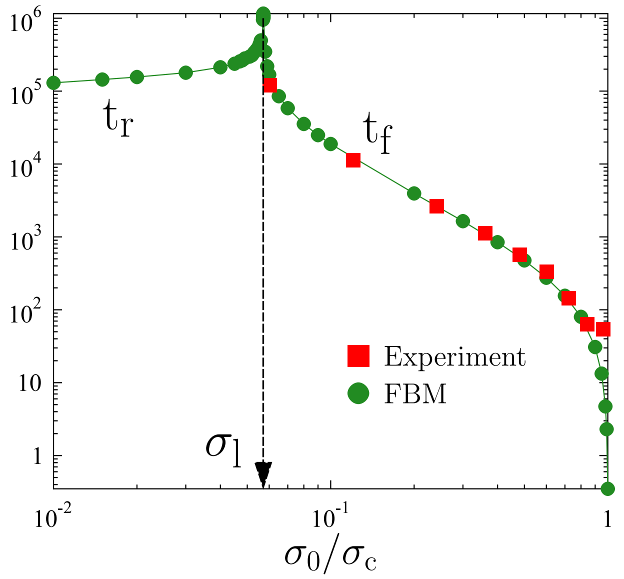

Figure 2. The relaxation time t_r and lifetime t_f of the system as a function of the external load. The threshold load σ_l below which no macroscopic failure occurs is indicated by the vertical dashed line. The numerical solution of the full model including also healing provides an excellent fit of the lifetime data for asphalt specimens [25, 26] above the threshold load σ_l.

loading cycles as a function of the number of cycles N_{cycle}. Note that in our model there is practically no difference between loading the system at a constant load σ_0 and periodic loading with a constant amplitude equal to σ_0. For the quantitative comparison we considered a Weibull distribution for the breaking thresholds

$$P(x) = 1 - \exp\left[-\left(x/\lambda_b\right)^{m_b}\right], \tag{13}$$

where the index b denotes p and c for immediate breaking and damage, respectively. The excellent quantitative agreement of the experimental and theoretical results presented in figure 1(b) was obtained by varying solely three parameters a, γ, and τ, while the Weibull parameters were fixed. Figure 1(c) presents the verification of the scaling law on experimental data. The good quality data collapse obtained by rescaling the two axes demonstrates the validity of the scaling form equation (11). In figure 1(d) the scaling function of the experimental data is re-plotted as a function of the time-to-failure where a power law behavior is evidenced in agreement with the analytic prediction of equation (12).

We carried out computer simulations with the full model taking into account the effect of immediate breaking, damaging and healing, to determine the lifetime of the system as a function of the external load. Healing dominates if for a fixed load σ_0 the memory time τ is smaller than the lifetime obtained without healing $\tau \lesssim t_f(\sigma_0, \tau = +\infty)$. Computer simulations revealed that for low load values the damage accumulation becomes limited, and a threshold load σ_l emerges below which the system relaxes, i.e., the deformation $\varepsilon(t)$ converges to a limit value with a characteristic relaxation time t_r resulting in an infinite lifetime. Figure 2 presents the characteristic timescale of the system varying the external load over a broad range. Above σ_l the lifetime t_f of the system defines the

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characteristic time. It can be seen that the model provides an excellent agreement with the measured lifetime of asphalt samples for $\sigma_0 > \sigma_1$ recovering also the Basquin exponent $\alpha = 2.2 \pm 0.05$ [25]. In the low load regime $\sigma_0 < \sigma_1$; unfortunately, no experimental data set is available in the literature for comparison.

4. Crackling noise: bursts triggered by damage sequences

The macroscopic time evolution of the system presented above is characterized by the scaling laws of the deformation–time histories and by the Basquin law of rupture life. The main advantage of our fiber bundle model is that it makes it possible to investigate the underlying microscopic failure process, and how the macroscopic evolution emerges as a consequence of the competition of the two failure modes of fibers at the microscale.

Rewriting equation (6) in the form of the constitutive equation for simple FBMs as

$$\frac{\sigma_0}{1 - F(c(t))} = [1 - G(p(t))]\, p(t), \tag{14}$$

it can be seen that even if the external load σ_0 is constant, the slow damage process quasi-statically increases the load on the system: ageing fibers accumulate damage and break slowly one by one in the increasing order of their damage thresholds. After a number Δ_d of damage breakings, the emerging load increment on the remaining intact fibers may become sufficient to trigger a burst of immediate breakings. Since load redistribution and immediate breaking occur on a much shorter timescale than damage accumulation, the entire fatigue process can be viewed on the microlevel as a sequence of bursts of immediate breakings triggered by a series of damage events whose duration defines the waiting times T, i.e., the time intervals between the bursts. In the absence of healing, the unlimited damage accumulation on the left-hand side of equation (14) leads to macroscopic failure at any finite external load $\sigma_0 > 0$ which occurs in the form of a catastrophic burst of immediate breakings when the load of single fibers $p(t)$ reaches the critical value p_c of simple FBMs.

In order to analyze the complex bursting activity of the model in the framework of equal load sharing, we carried out computer simulations with finite samples of N fibers, where the fibers had uniformly distributed breaking thresholds between 0 and 1. When the external load σ_0 is imposed, the weak fibers with breaking thresholds $p_\mathrm{th}^i < p_0$ break immediately, where p_0 is obtained from the constitutive equation, equation (5), of static FBMs. Since each breaking event increases the load on single fibers $p(t)$, bursts of immediate breakings may be triggered by damage sequences. Under low external loads σ_0, the load increments after local breakings may not be sufficient to trigger bursts, so most of the fibers break in long damage sequences, whose size Δ_d is defined by the number of fibers breaking in the sequence. When σ_0 approaches the static fracture strength σ_c, the system becomes more sensitive to load increments of damage breakings resulting in an intense bursting activity. The size of bursts Δ is determined as the number of simultaneously breaking fibers as a consequence of a damage sequence. Figure 3 shows representative examples of the size of damage sequences Δ_d and bursts Δ, and, furthermore, the waiting times between bursts T for a series of individual events obtained at $\sigma_0/\sigma_\mathrm{c} = 0.07$ for a system of $N = 10^5$ fibers. It can be observed in figure 3 that due to the disorder of failure thresholds of fibers $p_\mathrm{th}^i, c_\mathrm{th}^i, i = 1, \ldots, N$, all the quantities have strong fluctuations.

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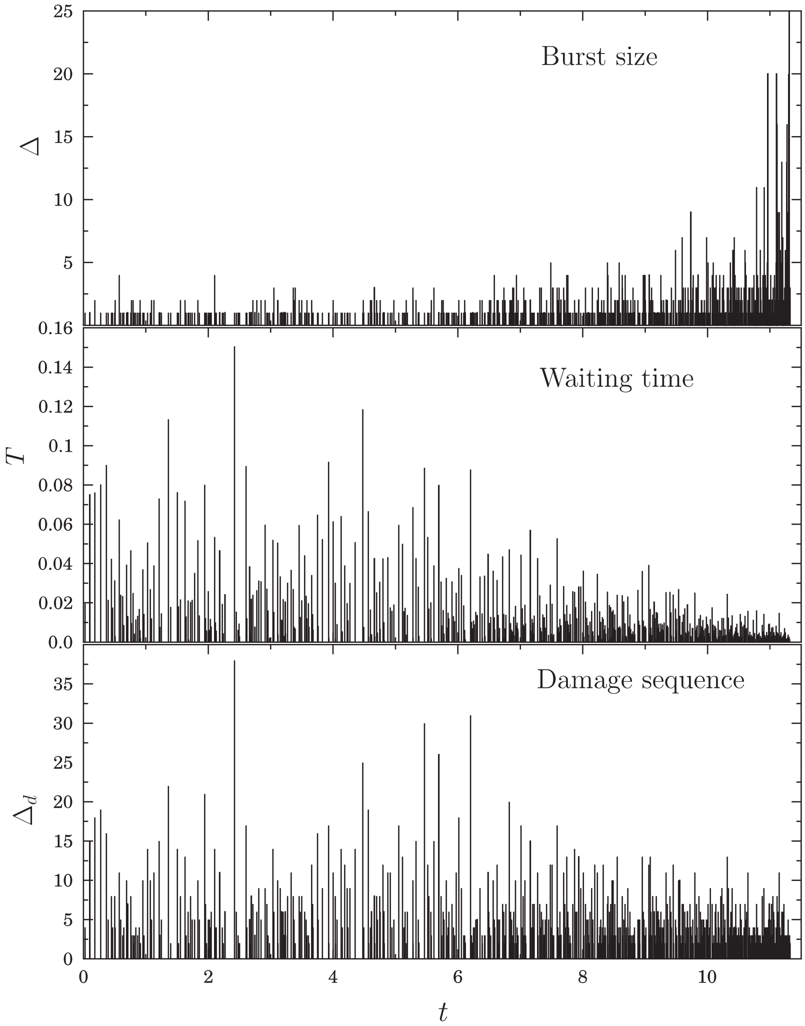

Figure 3. Bursting activity during the time evolution of a system of 10^5 fibers under a constant external load $\sigma_0/\sigma_c = 0.07$. The burst size Δ, the size of damage sequences Δ_d, and the waiting time T are presented as a function of time t elapsed since the bundle was subjected to load. After a number Δ_d of fibers break over the duration T, the resulting load increment becomes sufficient to trigger a burst of immediate breaking. The duration of damage sequences defines the waiting time between bursts. All the three quantities have strong fluctuations due to the disorder of fiber strength.

In spite of the smooth macroscopic response of the system, on the microlevel a jerky breaking sequence emerges. Since the fracture process accelerates as macroscopic failure is approached, the burst sizes Δ become larger, while the size of damage sequences Δ_d and their duration T get reduced in the vicinity of t_f.

The microscopic failure process is characterized by the size distribution of bursts $P(\Delta)$, damage sequences $P(\Delta_d)$, and by the distribution of waiting times $P(T)$. The burst size distributions $P(\Delta)$ are presented in figure 4 for several different load values σ_0. At small loads $\sigma_0 \ll \sigma_c$ most of the fibers break in long damage sequences, because the resulting load increments do not suffice to trigger bursts. In this case the system

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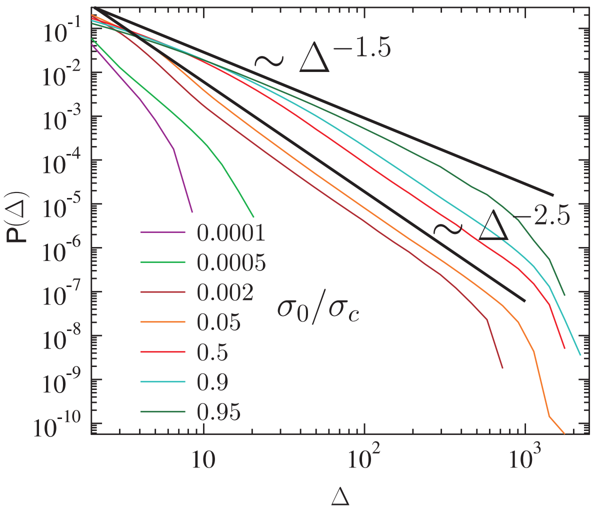

Figure 4. Size distribution of bursts obtained at different external loads σ_0. As σ_0 approaches the critical load σ_c a crossover occurs from a power law of exponent 5/2 to another one with a lower exponent 3/2.

behaves similarly to a simple FBM under quasi-static loading where the loading process was stopped far below σ_c. Consequently, the burst size distribution $P(\Delta)$ has a rapid exponential decay [17]–[22], [28, 29]. Increasing σ_0, the burst size distribution becomes a power law

$$P(\Delta) \sim \Delta^{-\xi}, \tag{15}$$

with the well-known ELS exponent of FBM $\xi = 5/2$ [17]–[22], [28, 29]. When macroscopic failure is approached, $\sigma_0 \to \sigma_c$, the failure process accelerates such that the size Δ_d and duration T of damage sequences decrease, while they trigger bursts of larger sizes Δ, and finally macroscopic failure occurs as a catastrophic burst of immediate failures. In the limiting case of $\sigma_0 \to \sigma_c$, a large number of weak fibers break in the initial burst; hence, the distribution $P(\Delta)$ becomes similar to that of quasi-static fiber bundles where the disorder distribution has a finite lower cutoff [28]–[30]. It can be seen in figure 4 that the burst size distribution exhibits a crossover to another power law with a lower exponent $\xi = 3/2$, in agreement with [28, 29]. The lower value of ξ indicates the dominance of large bursts in the limit of $\sigma_0 \to \sigma_c$.

Since damage events increase the load on the remaining intact fibers until an immediate breaking is triggered, the size of damage sequences Δ_d is independent of the damage characteristics $c(t)$ and $F(c_{th})$ of the material; instead, it is determined by the load bearing strength distribution $G(p_{th})$ of fibers. The size distribution of damage sequences can be determined by calculating the average number of fibers which have to be removed randomly from a fiber bundle under a constant load to trigger an avalanche, and then integrating it over the entire loading history. Analytic calculations and computer simulations revealed that the size distribution of damage sequences has a power law form with an exponential cutoff

$$P(\Delta_d) \sim \Delta_d^{-\omega} e^{-\Delta_d/\langle\Delta_d\rangle}, \tag{16}$$

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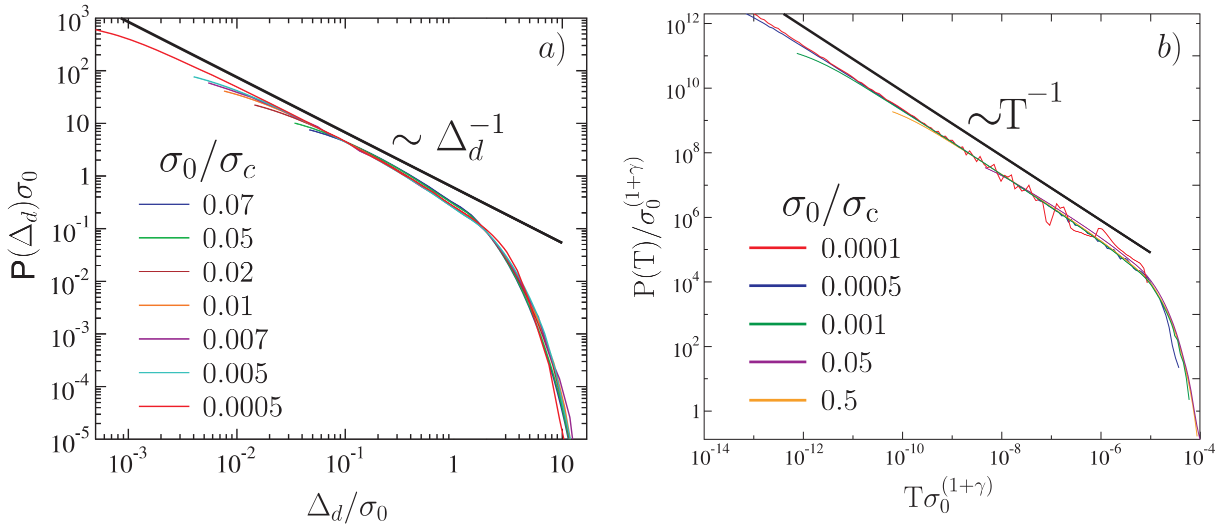

Figure 5. (a) Scaling plot of the size distribution of damage sequences. (b) Scaling plot of waiting time distributions obtained at different load values. Only the cutoff of the distributions depends on the details of the damage accumulation law.

where the value of the exponent $\omega = 1$ is independent of the disorder distribution and of the parameters a, γ characterizing the rate of damage accumulation. The average size $\langle \Delta_{\rm d} \rangle$ of damage sequences determining the cutoff of the size distribution $P(\Delta_{\rm d})$ decreases as a power law of the external load:

$$\langle \Delta_{\rm d} \rangle \sim \sigma_0^{-1}. \tag{17}$$

Figure 5(a) presents the scaling plot of distributions $P(\Delta_{\rm d})$ obtained at different external loads σ_0. The good quality data collapse verifies the validity of the scaling form equations (16) and (17). The damage law $c(t)$ of the material controls the timescale of the process of fatigue fracture through the temporal sequence of single damage events. In damage sequences, fibers break in the increasing order of their damage thresholds $c_{\rm th}^i$, which determine the time intervals Δt between consecutive fiber breakings. Hence, the distribution of inter-event times $P(\Delta t)$ can be obtained analytically on the basis of the property that the probability distribution of the ith largest element of a sorted sequence of N thresholds is sharply peaked for each i for large enough N values. It follows from the derivations that $P(\Delta t)$ has an explicit dependence on γ as

$$P(\Delta t) \sim \Delta t^{-(1-1/\gamma)}. \tag{18}$$

However, the durations of sequences $T = \sum_{j=1}^{\Delta_{\rm d}} \Delta t_j$, i.e., the waiting times between bursts, follow a universal power law distribution

$$P(T) \sim T^{-z} {\rm e}^{-T/\langle T \rangle}, \tag{19}$$

where the value of the exponent $z = 1$ does not depend on any details of the system and only the cutoff depends on γ:

$$\langle T \rangle \sim \sigma_0^{-(1+\gamma)}. \tag{20}$$

In figure 5(b) the scaling plots of waiting time distributions obtained at different load values are presented. The high quality data collapse obtained by rescaling the two axes, and the power law behavior over six orders of magnitude demonstrate the validity of the functional form equations (19) and (20).

5. Effect of stress concentration

The main advantage of the equal load sharing approach studied up to now is that it makes it possible to obtain analytic results for the most important characteristic quantities of the system, and it allows for large scale computer simulations. Since all intact fibers share the same load, the equal load sharing approach cannot capture stress inhomogeneities; the spatial sequence of local breakings is fully random in the fiber bundle. However, in realistic situations, when materials get damaged, the stress field becomes strongly inhomogeneous. It is an important question how the stress concentration arising around failed regions influences the microscopic failure process and the bursting activity.

In order to clarify this problem, we carried out computer simulations putting the fiber bundle on a square lattice of size L. After each breaking event the load of the failed fiber is redistributed equally over its intact nearest neighbors, i.e. at most over four fibers. Consequently, a considerable stress concentration arises along the surface of failed regions resulting in non-trivial spatial correlations. Since the increased load around broken fibers enhances both the rate of damage accumulation and the probability of immediate breaking, additional fiber failures occur correlated such that bursts become spatially correlated, resulting in growing cracks.

Figure 6(a) presents the burst size distributions for a square lattice of fibers of size $L = 401$ obtained at different values of the load with the damage parameters $a = 0.01$, $\gamma = 2.0$. It can be observed that like for the case of global load sharing, a power law distribution occurs over several decades of burst sizes. The value of the exponent $\xi_{\mathrm{LLS}} = 1.8 \pm 0.05$ falls between the two exponents of the ELS limit. It is interesting to note that the power law develops even at relatively low load values $\sigma_0/\sigma_{\mathrm{c}} > 0.06$. Since the load is redistributed over the close vicinity of the broken fiber, bursts are formed by spatially correlated breaking events. This has the consequence that as the burst proceeds, the stress concentration increases along its perimeter. However, due to the relatively low external load, the system is able to tolerate even large spatially correlated bursts. As σ_0 approaches σ_{c} another power law regime of $P(\Delta)$ arises, i.e. for small avalanches the distribution has a high exponent $\xi_{\mathrm{LLS}} \approx 9/2$, which coincides with the exponent of quasi-static LLS fiber bundles [30, 31]. The crossover to the high value of the exponent shows the dominance of the immediate breaking in the failure process, when the system is not capable of tolerating large bursts.

The distribution of waiting times between consecutive bursts $P(T)$ exhibits also a power law behavior with an exponent $z_{\mathrm{LLS}} = 1.4 \pm 0.05$ (see figure 6(b)). The higher value of the LLS exponent with respect to its ELS counterpart implies that due to the stress concentration around broken clusters, the failure process gets faster and the large waiting times become less frequent in the system. Computer simulations performed with uniform and Weibull distributions varying the damage accumulation exponent γ revealed that the value of the waiting time exponent z_{LLS} depends neither on the damage exponent γ nor on the distribution of the two failure modes.

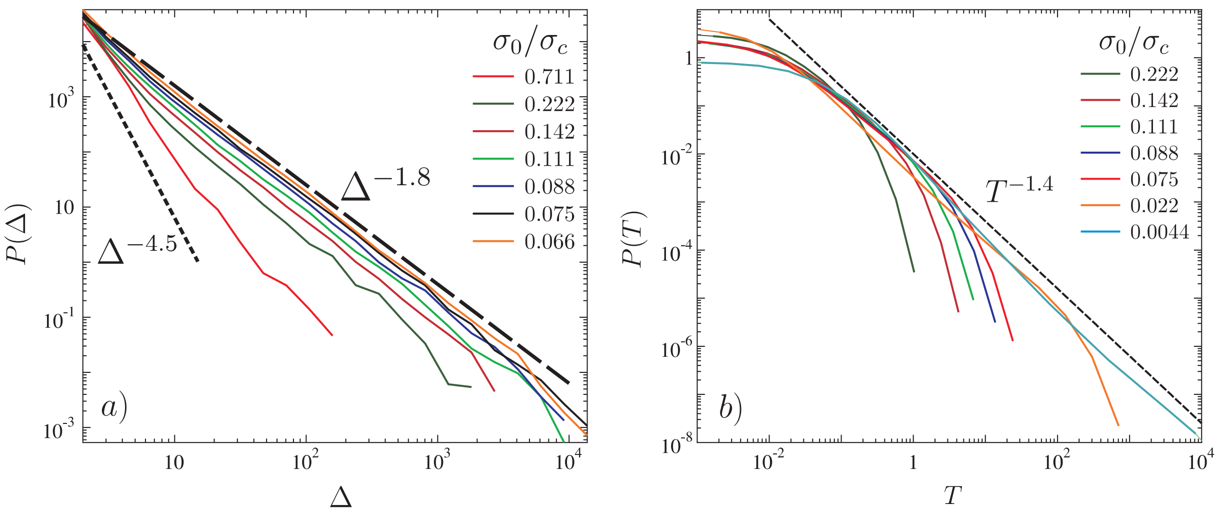

Figure 6. (a) Burst size distributions $P(\Delta)$ obtained at different external loads σ_0 under local load sharing conditions. The distributions follow a power law over a broad range of burst sizes with an exponent $\xi_{\mathrm{LLS}} = 1.8 \pm 0.05$. When σ_0 approaches the critical load σ_c for small avalanches, another power law regime develops whose exponent coincides with that of quasi-static LLS fiber bundles $\xi_{\mathrm{LLS}} = 9/2$ [30, 31]. (b) The waiting time distributions $P(T)$ show also a power law behavior with an exponent $z_{\mathrm{LLS}} = 1.4 \pm 0.05$ which is higher than its ELS counterpart.

6. Summary

We carried out a theoretical study of the sub-critical fracture of heterogeneous materials occurring under a constant external load focusing on the microscopic failure process. We presented an extension of the classical fiber bundle model to capture the basic ingredients of time dependent fracture. In the model, fibers fail due to two physical mechanisms, i.e. the fibers break due to their elastic response when the load on them exceeds the local strength, while intact fibers undergo an ageing process and break when the accumulated damage exceeds a random threshold. Our analytical calculations and computer simulations showed that the model provides a comprehensive description of damage enhanced time dependent failure. On the macrolevel we demonstrated that the deformation–time histories obtained at different external loads obey a generic scaling form, where the scaling function has a power law divergence as a function of time-to-failure. The model recovers the Basquin law of rupture life where the Basquin exponent coincides with the exponent of damage accumulation rate. We showed that healing of microcracks controls the failure process at low load levels determining the threshold load of the material below which the specimen suffers only a partial failure and has an infinite lifetime.

At the microlevel, the separation of timescales of the two failure mechanisms leads to a complex bursting activity. The slow damage process breaks the fibers one by one which gradually increases the load on the remaining intact fibers. As a consequence, the slow damage sequences trigger bursts of immediate breakings which can be recorded in

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the form of crackling noise in experiments. Due to the disorder of fiber strength, the size of damage sequences, the size of bursts, and the waiting times between them, have strong fluctuations, and can be characterized by probability distributions. Analytical and numerical calculations revealed that in the case of equal load sharing of fibers the distributions of the characteristic quantities of bursts have universal power law functional forms, where only the cutoff values depend on the details of the system. In the more realistic situation of localized load redistribution we found that the distributions of burst characteristics remain power laws; however, the LLS exponents are different from the corresponding ELS values. Due to the concentration of stress along the crack boundaries, the rupture proceeds faster, giving rise to a higher value of the waiting time exponent compared to the ELS case.

We demonstrated that our theoretical results provide a comprehensive description of the macroscopic time evolution of asphalt specimens measured during periodic loading at a constant amplitude. Unfortunately, in these asphalt experiments no acoustic noise was recorded due to technical difficulties. Recent experiments on various types of fracture processes revealed a power law distribution of waiting times between consecutive local breaking events [11], [32]–[35]. The measured value of the waiting time exponent typically falls between 1.0 and 2.0 in reasonable agreement with our model calculations. We propose further experimental tests of our theoretical predictions.

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