Fractal frontiers of bursts and cracks in a fiber bundle model of creep rupture

Zsuzsa Danku and Ferenc Kun*
Department of Theoretical Physics, University of Debrecen, P.O. Box 5, H-4010 Debrecen, Hungary

Hans J. Herrmann
Computational Physics IfB, HIF, ETH, Hönggerberg, 8093 Zürich, Switzerland
and Departamento de Fisica, Universidade Federal do Ceara, 60451-970 Fortaleza, Ceara, Brazil
(Received 8 October 2015; published 2 December 2015)

We investigate the geometrical structure of breaking bursts generated during the creep rupture of heterogeneous materials. Using a fiber bundle model with localized load sharing we show that bursts are compact geometrical objects; however, their external frontiers have a fractal structure which reflects their growth dynamics. The perimeter fractal dimension of bursts proved to have the universal value 1.25 independent of the external load and of the amount of disorder in the system. We conjecture that according to their geometrical features, breaking bursts fall in the universality class of loop-erased self-avoiding random walks with perimeter fractal dimension 5/4 similar to the avalanches of Abelian sand pile models. The fractal dimension of the growing crack front along which bursts occur proved to increase from 1 to 1.25 as bursts gradually cover the entire front.

DOI: 10.1103/PhysRevE.92.062402 PACS number(s): 46.50.+a, 89.75.Da, 05.90.+m

I. INTRODUCTION

The fracture of heterogeneous materials proceeds in bursts generated by newly nucleating cracks or by intermittent propagation steps of crack fronts [1–4]. Measuring acoustic emissions of bursts the fracture process can be decomposed into a time series of crackling events [5]. The analysis of the statistics of crackling noise provides valuable insight into the dynamics of fracture making it also possible to design methods to forecast the imminent catastrophic failure [6]. Recent investigations have revealed that beyond the time evolution of crackling time series, the temporal dynamics of single bursts encodes also interesting information about the presence and nature of correlations in the underlying stochastic process [7,8]. Less is known, however, about the spatial evolution of bursts and their geometrical structure.

Individual bursts of breaking have been observed experimentally during the propagation of a planar crack. In the experiments of Refs. [2,9] a weak interface between two sintered plastic plates was created introducing also disorder in a controlled way by sand blasting the surface of the plates. The loading and boundary conditions ensured that a single crack emerged constrained to a plane. Bursts were identified as sudden local jumps of the front whose spatial and temporal dynamics could be studied by means of high-speed imaging. Detailed analysis revealed that bursts are composed of extended clusters which are nearly compact objects with an anisotropic shape, i.e., they are elongated along the front. The scaling exponent of the two side lengths of the bounding box of clusters was found to be related to the roughness exponent of the crack front [9].

The experimental findings on the statistical and geometrical features of bursts could be explained by the crack line model where the long-range elastic interaction proved to be essential [10,11]. Another interesting approach to the problem was presented in Refs. [12–14], where the interface between a stiff and a soft solid block was discretized in terms of fibers on a square lattice. While slowly increasing the external load, a single growing crack with a straight average profile was obtained by introducing a gradient for the strength of fibers. It was shown that when crack propagation is controlled by gradient percolation the crack frontier is fractal with a perimeter dimension consistent with the hull exponent $7/4$ of percolation clusters.

Here we present a theoretical investigation of the geometrical structure of breaking bursts driven by the short-range redistribution of stress. We consider a fiber bundle model (FBM) of creep rupture where the localized load sharing (LLS) following the breaking of fibers leads to the emergence of a single propagating crack. In the model the system evolves under a constant external load, where sudden bursts are triggered along a propagating front by slow damaging. The model is well suited to study single bursts because creep does not affect the dynamics of breaking avalanches; however, it allows significantly larger avalanche sizes compared to fracture processes occurring under a quasistatically increasing external load in FBMs. We show by means of computer simulations that due to the short-range interaction, bursts are fully connected compact geometrical objects with nearly isotropic shapes. However, the external frontier of a burst proved to be fractal with a dimension independent of the load and of the degree of disorder. We argue that avalanches driven by the strongly localized redistribution of load fall in the universality class of loop erased self-avoiding random walks with the perimeter dimension $5/4$.

II. BURSTS IN A FIBER BUNDLE MODEL OF CREEP RUPTURE

To investigate the geometry of bursts we use a fiber bundle model which has been introduced recently for the creep rupture of heterogeneous materials [15–17]. The sample is represented by a parallel set of fibers which are organized on a square lattice of side length L. The fibers have linearly elastic behavior with a constant Young modulus E. Subjecting the bundle to a constant

*ferenc.kun@science.unideb.hu

load σ_0 below the fracture strength σ_c of the system, the fibers break due to two physical mechanisms: immediate breaking occurs when the local load σ_i on fibers exceeds their fracture strength $\sigma_{\rm th}^i$, $i = 1,\ldots,N$, where $N = L^2$ is the number of fibers. Under a subcritical load $\sigma_0 < \sigma_c$ this breaking mechanism would lead to a partially failed configuration with an infinite lifetime. Time dependence arises such that those fibers, which remained intact under a given load, undergo an aging process accumulating damage $c(t)$. We assume that the damage accumulation Δc_i per time step Δt has a power-law dependence on the local load

$$\Delta c_i = a\sigma_i^{\gamma}\Delta t, \qquad (1)$$

where a is a constant and the exponent γ controls the characteristic time scale of the aging process with $0 \leqslant \gamma < +\infty$. The total amount of damage $c_i(t)$ accumulated up to time t can be obtained by integrating over the entire loading history of fibers $c_i(t) = a\int_0^t \sigma_i(t')^{\gamma}dt'$. Fibers can sustain only a finite amount of damage so that when $c_i(t)$ exceeds the local damage threshold $c_{\rm th}^i$ the fiber breaks. After each breaking event the load of the failed fiber gets redistributed over the remaining intact ones. We assume localized load sharing, i.e., after failure events the load of broken fibers is equally redistributed over their intact nearest neighbors on the lattice. When a fiber without intact nearest neighbors breaks, the extension of the neighborhood is gradually increased in steps of one lattice site, until the neighborhood contains at least one intact fiber. Here such situation mainly occurs inside the last, catastrophic burst which does not affect the statistics of the data.

Due to the wide separation of the characteristic time scales between the slow damage process and immediate breaking a highly complex time evolution emerges: damage breaking events gradually increase the load on the intact fibers in their vicinity which in turn can trigger sudden bursts of immediate breaking [16,18]. Eventually, the time evolution of creep rupture sets in as a series of sudden bursts separated by quiet periods of slow damaging.

Heterogeneity has two sources in the model, i.e., structural disorder of materials is represented by the randomness of breaking thresholds $\sigma_i^{\rm th}, c_i^{\rm th}$, $i = 1,\ldots,N$ of the two breaking modes, while additional disorder is induced by the heterogeneous stress field generated by the short-ranged load redistribution around failed regions. For both threshold values we assume uniform distributions over an interval $[1-\delta_x, 1+\delta_x]$, where x stands for $c_{\rm th}$ and $\sigma_{\rm th}$. Tuning the width of the distributions δ_x one can control the amount of threshold disorder in the system. To promote the effect of stress concentration on the overall evolution of cracks a small amount of disorder is considered for damage $\delta_{c_{\rm th}} = 0.2$ with the exponent $\gamma = 5$. Simulations were carried out on a square lattice of linear size $L = 401$ varying the amount of threshold disorder $\delta_{\sigma_{\rm th}}$ of immediate breaking in the range $0.2 \leqslant \delta_{\sigma_{\rm th}} \leqslant 0.5$. The model has been successfully applied to describe the time evolution of damage-induced creep rupture [15,18,19], the statistics of crackling bursts [16,17,20], and even the average temporal profile of bursts [8]. In the present study we focus on the geometrical features of bursts and of the crack front.

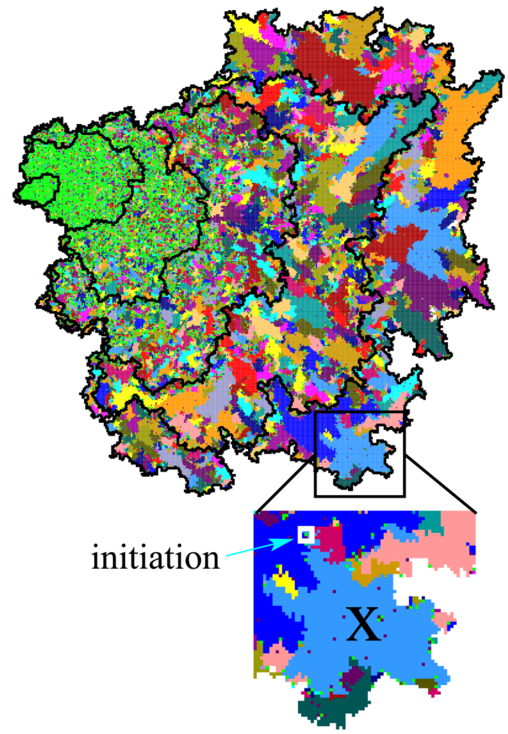

FIG. 1. (Color online) Evolution of a single crack until macroscopic failure of the system at the external load $\sigma_0/\sigma_c = 0.01$. The crack starts at the top left corner with a large amount of damage breaking (green area). Damage sequences trigger bursts (spots with randomly assigned colors) which occur at the propagating crack front. The bold black lines highlight the position of the crack front at several times during the growth process. A magnified view of a small portion of the crack in the black square is presented where the burst of light blue color (with a cross in the middle) was initiated by the damage breaking in the white square.

III. STRUCTURE OF SINGLE BURSTS

The low damage disorder and localized load sharing ensure that at any subcritical external load a single growing crack emerges which gradually evolves through bursts leading to global failure of the system in a finite time. Figure 1 presents a representative example of a growing crack where individual avalanches of immediate breakings can also be distinguished. It can be seen that early stages of the crack growth are dominated by damage-induced breaking of fibers indicated by the large green area. However, as the crack reaches larger sizes the high load accumulated along the front leads to triggering larger and larger bursts (spots of randomly assigned color). Bursts are characterized by their size Δ which is the number of fibers failing in the cascade of immediate breakings. Bursts are always induced by damage sequences; however, in later stages of the evolution a few or even a single damage breaking can be sufficient to trigger large bursts. The green dots scattered along the perimeter of extended spots of other colors correspond to these damage sequences in the figure. Note that the crack propagation does not have a preferred direction, since no gradient is imposed on stress, strain, or strength of fibers. Hence, the average front position does not follow a

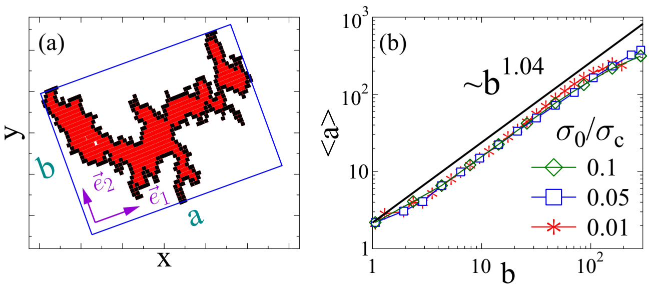

FIG. 2. (Color online) (a) Definition of the bounding box of a burst. The burst perimeter, highlighted by a bold black line, is formed by those sites of the broken cluster which have less then four broken neighbors. (b) Longer side of the bounding box a as a function of the shorter one b in a log-log plot.

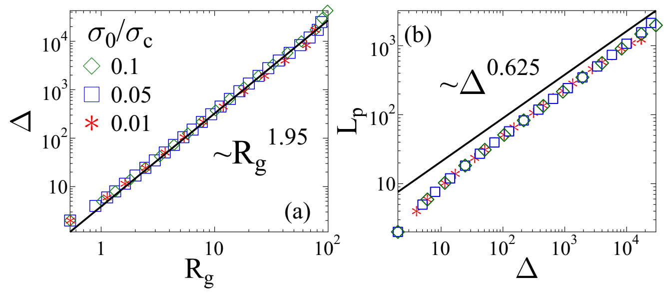

FIG. 3. (Color online) (a) Burst size Δ as a function of the radius of gyration R_g for three values of the external load. (b) Perimeter of bursts as a function of their area Δ.

straight line, instead it gets curved as the crack reaches larger sizes.

It can be observed in Fig. 1 that from a geometrical point of view single bursts are compact objects, i.e., the localized load sharing ensures that they are dense and they practically do not contain islands of intact fibers. However, due to the interplay of the disordered strength and of the heterogeneous stress field, bursts have a complex external frontier. To characterize the overall geometry of bursts we determined the eigenvectors $\vec{e}_1$, $\vec{e}_2$ of the tensor of inertia of single bursts. Then we constructed the bounding box of bursts as the rectangle along the vectors $\vec{e}_1, \vec{e}_2$ fully surrounding the cluster as it is illustrated in Fig. 2(a) where the longer and shorter edges are denoted by a and b, respectively. For the quantitative characterization of the shape of bursts we evaluated the average length of the longer edge $\langle a \rangle$ as a function of the shorter one b which is presented in Fig. 2(b) for three different load values σ_0/σ_c. A power-law relation of slope very close to unity is obtained showing that the side lengths are simply proportional to each other and the aspect ratio a/b does not depend on the burst size Δ. When long-range elastic interaction is taken into account bursts get elongated along the front resulting in anisotropic shapes [11,13,14].

Based on the nearly isotropic shape, we determined the radius of gyration R_g of bursts

$$R_g^2 = \frac{1}{N}\sum_{i=1}^{\Delta}(\vec{r}_i - \vec{r}_c)^2, \qquad (2)$$

where $\vec{r}_i$ $(i = 1, \ldots, N)$ denotes the position of broken fibers of the burst and $\vec{r}_c$ is the center of mass of the cluster. The value of R_g was averaged over bursts of equal size Δ accumulating all events up to failure. In Fig. 3(a) the burst size Δ, i.e., the area of the cluster of broken fibers, is plotted as a function of R_g where a power-law functional form is obtained

$$\Delta \sim R_g^D. \qquad (3)$$

The value of the exponent is $D \approx 2$ which shows the high degree of compactness of bursts. Note that the statistics of the data is mainly determined by the size distribution of bursts. Recently, we have shown that the size of bursts Δ is power-law distributed with an exponential cutoff controlled by the external load [8,16]. Fluctuations are kept low in the figures by simulating an ensemble of 10 000 samples for each parameter set. The $\Delta(R_g)$ curves obtained at different loads σ_0 fall on top of each other, practically the only effect of σ_0 is that it controls the upper cutoff of Δ since at lower loads smaller bursts are triggered [8,17].

The most remarkable feature of avalanches is that they have a tortuous frontiers composed of a large number of valleys and hills (see Fig. 1 and Fig. 2). The structure of the burst perimeter is determined by the interplay of the disorder of the failure thresholds σ_{th} and of the stress field during the growth of the avalanche. The avalanche usually starts from a single breaking fiber. As the load gets redistributed some of the surrounding intact fibers may exceed their failure threshold and break, leading to an expansion of the burst. Bursts proceed through such subsequent breaking and load redistribution steps until all intact fibers along the burst frontier can sustain the elevated load. For a quantitative assessment we determined the number of perimeter sites L_p of each burst as a function of the burst size Δ. On the square lattice those broken fibers are identified as perimeter sites which have less than four broken neighbors in the same cluster. Inside of the bursts a very small amount of intact fibers may remain isolated, they are removed before perimeter identification. In Fig. 2(a) the perimeter is highlighted by a bold black line while the bulk of the cluster is red.

The perimeter length L_p averaged over all bursts of a fixed size Δ is presented in Fig. 3(b) as a function of Δ. In the regime of large burst sizes a power law is evidenced

$$L_p \sim \Delta^{\xi}, \qquad (4)$$

where the value of the exponent ξ was obtained as $\xi = 0.625 \pm 0.015$. This feature implies that the avalanche frontier is a fractal with fractal dimension $D_f = \xi D = 1.25 \pm 0.03$, where $D = 2$ was substituted. It is important to emphasize that the fractal dimension D_f proved to be universal, i.e., it depends on neither the external load σ_0 nor any details of damage accumulation such as γ, as long as single crack propagation is ensured in a heterogeneous environment. The only role of the damage mechanism is that fibers breaking due to damage initiate the bursts. Once the burst has started it is driven by the gradual redistribution of load. Figure 4 presents the perimeter length of avalanches as a function of the radius of gyration for several different values of the width $\delta_{\sigma_{\text{th}}}$ of the threshold distribution of immediate breaking. It can be observed that the curves fall on top of each other, which indicates that the amount of disorder does not have a noticeable effect until its value is

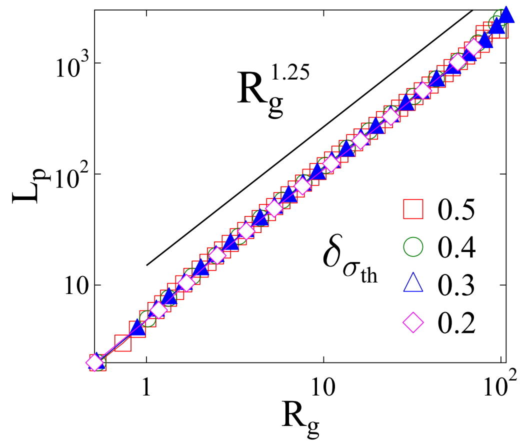

FIG. 4. (Color online) Perimeter length L_p of bursts as a function of the radius of gyration R_g for four different amounts of threshold disorder $\delta_{\sigma_{th}}$. All the curves fall on top of each other.

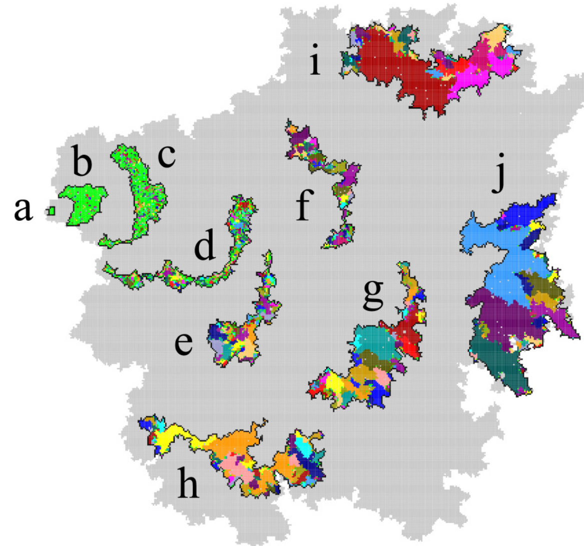

FIG. 5. (Color online) Clusters of bursts inside the growing crack of Fig. 1. The same color coding is used as in Fig. 1, i.e., the damage breakings which trigger bursts are in green, while the bursts have randomly assigned colors different from green. The clusters denoted by $a, \dots, j$ are generated at different stages of crack growth, where a indicates the initial damage cluster without any burst. The other clusters correspond to different ratios L_p^d/L_p of the perimeter length L_p and of the number of damage breakings in the perimeter L_p^d: (b) $0.8-0.9$, (c) $0.5-0.6$, (d) $0.4-0.5$, (e) $0.2-0.3$, (f) $0.2-0.3$, (g) $0.1-0.2$, (h), (i), (j) $0-0.1$.

sufficiently high. The structure of the external frontier of bursts emerges as an outcome of the interplay of the short-range stress redistribution and of the strength disorder of fibers.

It is interesting to note that avalanches of other types of systems with short-range interaction show a striking similarity to the bursts of our LLS FBM. The Abelian sandpile model (ASM) is a paradigmatic model of self-organized criticality [21,22] where topling (overstressed) lattice sites relax by redistributing sand grains over their local neighborhood. Majumdar showed that the frontiers of avalanches driven by short-range redistribution in ASM can exactly be mapped to loop erased random walks (LERW) in two dimensions, and determined their fractal dimension exactly $D_f = 5/4$ [21]. Based on the growth dynamics, geometrical features, and robustness of the fractal dimension of burst perimeters we conjecture that breaking avalanches of FBMs fall in the universality class of LERWs similar to avalanches of granular piles [21].

IV. STRUCTURE OF THE PROPAGATING FRONT

Burst are always initiated along the propagating crack front where the load redistribution after damage breaking can give rise to immediate breaking of fibers. It can be observed in Fig. 1 that the early stage of crack growth is dominated by the damage mechanism: the crack size has to reach a threshold size to achieve a sufficiently high stress concentration at the crack frontier to trigger bursts. As the crack grows all the load kept by the fibers inside the crack area accumulates at the crack front which gives rise to larger and larger bursts. The localized load sharing ensures that the crack is a dense set of broken fibers and there are no broken clusters ahead of the front (no process zone can form). It can be observed in Fig. 1 that when damage breaking dominates at the early stage of crack growth a smooth front is formed, while it gets tortuous as burst gradually overtake the control of propagation. It follows that the key feature which determines the degree of smoothness of the front is the ratio between the damage and immediate breakings along the perimeter.

To quantify the change of structure of the crack front, inside the crack we identified clusters of broken fibers formed by the burst and by the damage breakings which occurred within certain time intervals. Such clusters are highlighted in Fig. 5

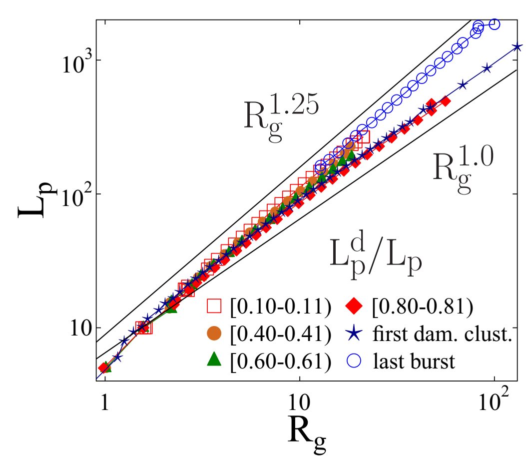

FIG. 6. (Color online) Perimeter length of clusters of bursts inside the crack for several values of the ratio L_p^d/L_p of the number of damage breakings L_p^d along the cluster perimeter and of the perimeter length L_p. For completeness, the initial cluster of damage breakings (*first dam. clust.*) and the last burst of the crack are also included. Damage breaking favors a smooth cluster boundary which implies effectively $D_f = 1$. As the fraction L_p^d/L_p decreases, bursts determine the cluster perimeter so that a gradual crossover occurs to $D_f = 1.25$.

inside the crack of Fig. 1. We determined the total perimeter length L_p of the clusters and the number of damage breakings L_p^d contained in L_p. Then we grouped the clusters according to the value of the ratio $L_p^d/L_p \leqslant 1$ using 0.01 for the bin size. Figure 6 presents the perimeter L_p of clusters for a few selected bins of L_p^d/L_p as a function of the radius of gyration R_g. It can be observed in the figure that when damage breaking dominates $L_p^d/L_p \approx 1$ along the external frontier of clusters, the perimeter fractal dimension takes the effective value $D_f = 1.0$, as it is expected for smooth lines. However, as the fraction L_p^d/L_p decreases, immediate breakings of bursts control more the geometry of the perimeter so that a gradual crossover is obtained to the higher value $D_f = 1.25$ of the fractal dimension of burst perimeters. It follows that in our localized load sharing fiber bundle model of creep rupture the fractal dimension of the propagating crack front depends on the stage of propagation at which it is measured.

V. DISCUSSION

We investigated the geometrical structure of breaking bursts and of the crack front in the framework of a fiber bundle model with localized load sharing. The model has been developed for the creep rupture of heterogeneous materials under constant subcritical external loads. In the model, fibers break due to two reasons: When the local load surpasses the strength of a fiber immediate breaking occurs. Time dependence is introduced by the second breaking mechanism, i.e., loaded fibers accumulate damage and break when the amount of damage exceeds a respective threshold. Damaging fibers trigger bursts of immediate breakings which occur along the front of a growing crack as sudden local advancements of the front. The bursting dynamics is mainly controlled by the localized redistribution of load after fiber failures (short-range interaction) and by the strength disorder.

We showed that due to localized load sharing, single bursts form compact geometrical objects of a nearly isotropic shape. However, the external frontier of bursts has a fractal structure characterized by the perimeter fractal dimension $D_f = 1.25$. Computer simulations revealed that the value of the fractal dimension is universal, i.e., it does not depend on either the external load or the amount of disorder in the range considered. The main role of the external load is that it controls the characteristic time scale of the failure process and the number and cutoff size of bursts that occur up to macroscopic failure, while the precise amount of disorder does not have a crucial effect until it is sufficiently high to prevent sudden early collapse (brittle failure) of the system.

Based on the value of the perimeter fractal dimension and on its striking universality, we conjecture that crackling bursts controlled by the localized redistribution of load fall in the universality class of loop erased self-avoiding random walks similarly to the avalanches of Abelian sand pile models [21,22]. When long-range elastic interaction along the front dominates the crack propagation, different geometrical features are found [9].

Our simulations showed that the geometrical structure of the crack front changes as the system evolves: damage breakings favor a smooth front while the bursts tend to make it more tortuous. The key parameter to control the front geometry proved to be the fraction of damage breakings along the front. For high values of the damage fraction, corresponding to the early stage of crack growth, the fractal dimension tends to 1, while for low fractions, obtained at larger crack sizes, a gradual crossover occurs to the burst perimeter dimension 1.25.

Compared to simple fiber bundle models of fracture with quasistatic loading, the creep dynamics has the advantage that a single growing crack emerges under a constant external load and large bursts develop along the crack front. However, since no gradient is imposed on stress, strain, or strength of fibers the crack front is curved. This implies that no average front position can be defined so that we cannot study the roughness of the front.

In our analysis the parameters of the damage mechanism, i.e., the γ exponent and the amount of disorder of the damage thresholds $\delta_{c_{\text{th}}}$ were set to ensure the dominance of stress concentration in the failure process which leads to the emergence of a single growing crack. However, for low exponent $\gamma \to 0$ and high disorder $\delta_{c_{\text{th}}} \to 1$ the breaking process gets disorder dominated where a large number of cracks grow simultaneously [17]. We note that the geometrical structure of bursts does not depend on the precise value of the damage parameters; however, from a numerical point of view it is advantageous to perform the analysis in the phase of single crack growth because here bursts and cracks reach larger sizes.

ACKNOWLEDGMENTS

We thank the projects TAMOP-4.2.2.A-11/1/KONV-2012-0036, TAMOP-4.2.2/B-10/1-2010-0024, OTKA K84157, and ERANET_HU_09-1-2011-0002 and ERC Advanced Grant 319968 FlowCCS.

[1] J. P. Sethna, K. A. Dahmen, and C. R. Meyers, Nature **410**, 242 (2001).
[2] K. J. Maloy, S. Santucci, J. Schmittbuhl, and R. Toussaint, Phys. Rev. Lett. **96**, 045501 (2006).
[3] M. Stojanova, S. Santucci, L. Vanel, and O. Ramos, Phys. Rev. Lett. **112**, 115502 (2014).
[4] J. Rosti, X. Illa, J. Koivisto, and M. J. Alava, J. Phys. D **42**, 214013 (2009).
[5] J. Baró, A. Corral, X. Illa, A. Planes, E. K. H. Salje, W. Schranz, D. E. Soto-Parra, and E. Vives, Phys. Rev. Lett. **110**, 088702 (2013).
[6] J. Vasseur, F. B. Wadsworth, Y. Lavalle, A. F. Bell, I. G. Main, and D. B. Dingwell, Sci. Rep. **5**, 13259 (2015).
[7] L. Laurson, X. Illa, S. Santucci, K. Tore Tallakstad, K. J. Maloy, and M. J. Alava, Nat. Commun. **4**, 4927 (2013).
[8] Z. Danku and F. Kun, Phys. Rev. Lett. **111**, 084302 (2013).
[9] K. T. Tallakstad, R. Toussaint, S. Santucci, J. Schmittbuhl, and K. J. Måløy, Phys. Rev. E **83**, 046108 (2011).
[10] D. Bonamy, S. Santucci, and L. Ponson, Phys. Rev. Lett. **101**, 045501 (2008).
[11] L. Laurson, S. Santucci, and S. Zapperi, Phys. Rev. E **81**, 046116 (2010).

[12] S. Pradhan, A. Hansen, and B. K. Chakrabarti, Rev. Mod. Phys. **82**, 499 (2010).

[13] K. S. Gjerden, A. Stormo, and A. Hansen, Phys. Rev. Lett. **111**, 135502 (2013).

[14] K. S. Gjerden, A. Stormo, and A. Hansen, Frontiers Phys. **2**, 66 (2014).

[15] F. Kun, Z. Halász, J. S. Andrade, and H. J. Herrmann, J. Stat. Mech. (2009) P01021.

[16] F. Kun, H. A. Carmona, J. S. Andrade, Jr., and H. J. Herrmann, Phys. Rev. Lett. **100**, 094301 (2008).

[17] Z. Halász, Z. Danku, and F. Kun, Phys. Rev. E **85**, 016116 (2012).

[18] F. Kun, M. H. Costa, R. N. C. Filho, J. S. Andrade, Jr., J. B. Soares, and H. J. Herrmann, J. Stat. Mech. (2007) P02003.

[19] Z. Danku and F. Kun, Sci. Rep. **3**, 2688 (2013).

[20] Z. Danku and F. Kun, Frontiers Phys. **2**, 08 (2014).

[21] S. N. Majumdar, Phys. Rev. Lett. **68**, 2329 (1992).

[22] D. Dhar, Phys. Rev. Lett. **64**, 1613 (1990).