Part VII.

Broken gauge symmetry: Superconductivity

History: Discovered in 1911 by Kamerling Ones: Hg, $T_c = 4.2K$. Interestingly in the same experiment he observed the superfluid transition in He at 2.2K, but not recognized its importance.

Up to this moment the following Nobel prizes have been awarded for the subject:

- 1913: Kamerling Ones
- 1972: BCS theory
- 1973: Josephson effect
- 1987: Ceramic superconductors: Bednorz and Müller
- 2003: Abrikosov, Ginzburg, Leggett: Vortices, non-s wave supercond.

Superconductivity:

The Cooper problem: Let us have a non-interacting Fermi gas at $T=0$. We approach to this system 2 electrons that attract each other (why? we do not ask now). We are interested to find a localized state for the two electrons such to have for this "pair" state the minimum energy. See Fig 13, the two electrons with k_1, k_2 and $|k_1|, |k_2| > k_F$. As seen the role of the many-body system here is only to have $|k_1|, |k_2| > k_F$. If we write the problem in center of mass coordinates, we find for the coordinates of the centre of mass:

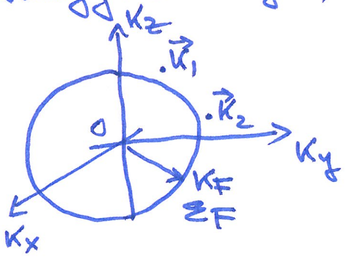

Fig 13: The Cooper problem for 2 electrons

$$\vec{R} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2);\ \vec{K} = \vec{k}_1 + \vec{k}_2;\ M = m_1 + m_2$$

(24) $$\hat{H}_R = -\frac{\nabla_R^2}{2M};\ (\hbar = 1);\ E_1 = \frac{K^2}{2M};\ \Psi_c(R) = \frac{1}{\sqrt{V}} e^{i\vec{K}\cdot\vec{R}}$$ and

$\hat{H}_R \Psi_c(\vec{R}) = E_1 \Psi_c(\vec{R})$; the eigenvalue equation

For the reduced particles coordinates one has:

$\vec{r} = \vec{r}_2 - \vec{r}_1$; $\vec{k} = \frac{1}{2}(\vec{k}_2 - \vec{k}_1)$; $m = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_e}{2}$ for the reduced mass

(25) $\hat{H}(\vec{r}) = -\frac{\nabla_r^2}{2m} + V(r)$; $\Psi_2(\vec{r}) = \sum_{\vec{k}} a_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}$; $a_{\vec{k}}, E_2$ being unknown quantities

$\hat{H}(\vec{r})\Psi_2(\vec{r}) = E_2 \Psi_2(\vec{r})$, the eigenvalue equation

But because $[\hat{H}(\vec{R}), \hat{H}(\vec{r})] = 0$ it results that for

(26) $$\hat{H} = \hat{H}(\vec{R}) + \hat{H}(\vec{r});\quad \hat{H}\Psi = E\Psi \text{ eigenvalue problem, one has}$$
$$E = E_1 + E_2;\ \Psi = \Psi_c(\vec{R}) \cdot \Psi_2(\vec{r})$$

Since $\Psi_c(\vec{R})$ is invariant under the permutation of the two electrons, and the electrons being fermions, the antisymmetry of Ψ at the permutation of the two particles must be provided by $\Psi_2(\vec{r})$:

(27) $$\Psi_2(\vec{r})\Big|_{\substack{\text{with the}\\ \text{2 particles}}} = -\Psi_2(\vec{r})\Big|_{\substack{\text{with particles}\\ \text{interchanged}}}$$

But, since the electrons have a spin, and interspin interaction is not considered

(28) $\Psi_2(\vec{r}) = \phi(\vec{r}_1, \vec{r}_2)\cdot \eta(\alpha_1, \alpha_2)$; where $\phi(\vec{r}_1, \vec{r}_2)$ is the part depending on the geometrical coordinates, and $\eta(\alpha_1, \alpha_2)$ is the part depending on the spin projections α_1 and α_2 of the two electrons. In order to satisfy (27) two possibilities are present, namely I: ϕ symmetric and η antisymmetric, or II: one has ϕ antisymmetric and η symmetric (under the permutation of the 2 particles).

→ Let us analyze first the spin dependent part of (28). As mentioned, one has two cases:

a.) Symmetrical case: In order to have the same sign after the two particle permutation the following situations can occur:

$$(29)\quad \begin{aligned} \eta_1^S &= \eta_1(\uparrow)\eta_2(\uparrow)\ ;\ S^z = 1 \\ \eta_2^S &= \frac{1}{\sqrt{2}}\left(\eta_1(\uparrow)\eta_2(\downarrow) + \eta_1(\downarrow)\eta_2(\uparrow)\right);\ S^z = 0; \\ \eta_3^S &= \eta_1(\downarrow)\eta_2(\downarrow)\ ;\ S^z = -1 \end{aligned}$$

Here $\eta_i(\sigma)$, $i = 1,2$; $\sigma = \uparrow,\downarrow$ are representing the spin parts of the electron i. As seen, one has a triplet corresponding to $S=1$, which has 3 possible S^z projections namely $S^z = -1, 0, +1$. Note that $\eta_i(\sigma_i)$ are orthonormate, which means

$$(30)\quad \int \eta_i(\sigma_i)\eta_j(\sigma_j)\,d\sigma = \delta_{ij}\delta_{\sigma_i\sigma_j}$$

b.) Asymmetric case: In this case sign change occurs after permutation, for which one has only 1 possibility:

$$(31)\quad \eta_1^a = \frac{1}{\sqrt{2}}\left(\eta_1(\uparrow)\eta_2(\downarrow) - \eta_1(\downarrow)\eta_2(\uparrow)\right);\ S^z = 0,\ \text{hence } S = 0.$$

This is a singlet case in which the spins there are oriented in opposite directions.

→ Now let us concentrate on the $\vec{r}_i$ dependent part of the wave function in (28). First one observes that, see (24, 26):

$$(32)\quad E = E_1 + E_2\ ;\ E_2 = \frac{K^2}{2M}\ ;\ \vec{K} = \vec{k}_1 + \vec{k}_2,$$

so in order to minimize E (see (33)) so in the pair opposite $\vec{k}$ values will be preferred.

$$(33)\quad E_2 = 0 \Rightarrow \vec{K} = 0 \Rightarrow \vec{k}_2 = -\vec{k}_1 = \vec{k}$$

(see (A11) from Appendix)

After this step, based on (25), we have

$$(34)\quad \phi(\vec{r}_1, \vec{r}_2) = \sum_{\vec{k}} a_{\vec{k}}\, e^{i\vec{k}(\vec{r}_2 - \vec{r}_1)},$$

and because one has a spherical Fermi surface, so $\vec{k} \to -\vec{k}$ will not modify the physics. Furthermore, the $\sum_{\vec{k}}$ is made for a symmetrical interval around the origine, namely $\sum_{\vec{k}}$ and $\sum_{-\vec{k}}$ are equivalent, it results $a_{\vec{k}} = a_{-\vec{k}}$, hence

$$(35)\quad \sum_{\vec{k}} a_{\vec{k}}\, e^{i\vec{k}(\vec{r}_2 - \vec{r}_1)} \underset{\vec{k} \to -\vec{k}}{=} \sum_{-\vec{k}} a_{-\vec{k}}\, e^{-i\vec{k}(\vec{r}_2 - \vec{r}_1)} \underset{a_{\vec{k}} = a_{-\vec{k}},\ \sum_{\vec{k}} = \sum_{-\vec{k}}}{=} \sum_{\vec{k}} a_{\vec{k}}\, e^{i\vec{k}(\vec{r}_1 - \vec{r}_2)}$$

As a consequence one has

$$(36)\quad \phi(r_1, r_2) = \phi(r_2, r_1):$$

which means that the geometrical coordinates part of (28) is symmetric. Hence one has

$$(37)\quad \phi^S(r_1, r_2) = \sum_{\vec{k}} a_{\vec{k}}\, e^{i\vec{k}\underbrace{(\vec{r}_2 - \vec{r}_1)}_{\vec{r}}} = \sum_{\vec{k}} a_{\vec{k}}\, e^{i\vec{k}(\vec{r}_1 - \vec{r}_2)} = \sum_{\vec{k}} a_{\vec{k}}\, e^{i\vec{k}\cdot\vec{r}}$$

Consequently, the pair wave function becomes of the form

$$(38)\quad \Psi_2(\vec{r}) = \phi^S(\vec{r}) \cdot \eta_1^a(\sigma_1, \sigma_2)\ ;\ \vec{r} = \vec{r}_1 - \vec{r}_2$$

From here is seen that the pair is formed in between electrons with opposite momentum, and opposite spin projection.

Now it turns out that the Cooper pair is formed in between two electrons placed in the states $(\vec{k},\vec{\sigma}) \leftrightarrow (-\vec{k},-\vec{\sigma})$. Since both the momentum $(\vec{p}=\hbar\vec{k}=m\frac{d\vec{r}}{dt})$, and the angular momentum $(\vec{L}=\vec{r}\times\vec{p}=\vec{r}\times(m\frac{d\vec{r}}{dt}))$ contain 1 time derivative (and the spin is also angular momentum) it turns out that at time reversal $t\to -t$, both $\vec{k}$ and $\vec{\sigma}$ change sign. But since the Cooper pair is a $(\vec{k},\vec{\sigma})\leftrightarrow(-\vec{k},-\vec{\sigma})$ pair, it remains invariant. Hence this Cooper pair, hence the superconducting phase based on it has time reversal symmetry. So if on the system something is acting which breaks the time reversal symmetry will damage the Cooper pairs, consequently the superconductivity. Such effects are given e.g. by external magnetic field (in the presence of $\vec{B}$, the $\vec{\sigma}$ and $-\vec{\sigma}$ states have different energies, so the time reversal symmetry is broken); or magnetic impurities.

Another observation is that if the Cooper pair is formed then

(39) $\langle \hat{c}^{+}_{k,\sigma}\hat{c}^{+}_{-k,-\sigma}\rangle \neq 0$. But this is not invariant to a gauge transformation $\hat{c}^{+}_{k\sigma}\to e^{i\theta}\hat{c}^{+}_{k\sigma}$. Hence at the transition to the superconducting state, the gauge symmetry will be broken.

Furthermore, if we analyze the Cooper pair on the Fermi sphere we observe that the pairing is given by 2 electrons with opposite spin placed on opposite positions on the Fermi sphere (see Fig 14).

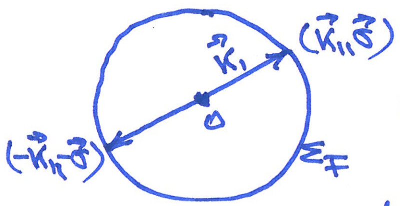

Fig 14: Placement of the Cooper pair on the Fermi sphere

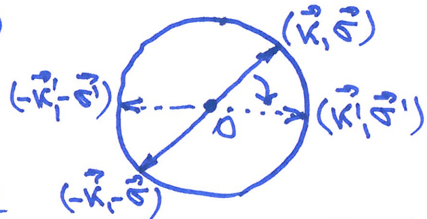

Fig 15: Scattering of a Cooper pair on the Fermi Sphere

One also observes that when a Cooper pair interacts with another Cooper pair (Fig 15), the opposite pair only moves to another position on the Fermi sphere, from $[(\vec{k},\vec{\sigma}),(-\vec{k},-\vec{\sigma})]$ to $[(\vec{k}',\vec{\sigma}'),(-\vec{k}',-\vec{\sigma}')]$.

Now the following question arrises: Do indeed the Cooper pair forms? (i.e. is it possible for $\Psi_2(\vec{r})$ in (38) to have the lowest eigenvalue in the bound state?) In order to answer this question, we must analyze the Schrödinger equation for the Hamiltonian in (25), namely for $\hat{H}(\vec{r})=\hat{H}_0+V(\vec{r})$; $\hat{H}_0=-\frac{\nabla^2_r}{2m}$; $m=\frac{m_e}{2}$ = reduced mass.

Since the spin part of (38) cancels out from the Schrödinger equation because in $\hat{H}$ nothing acts on the spin, one has

(40) $(\hat{H}_0+V(\vec{r}))\phi^s(\vec{r})=E_1\phi^s(\vec{r})$; where E_1 is the energy of the pair and one has $\phi^s(\vec{r})=\sum_{\vec{k}'} a_{\vec{k}'} e^{i\vec{k}'\vec{r}}$ (see (37)), where $a_{\vec{k}'}$ and E_1 are the unknowns of the problem.

Now introducing $\phi^s(\vec{r})$ in (40) and taking into account that

(41) $\hat{H}_0\phi^s(\vec{r})=\sum_{\vec{k}'} a_{\vec{k}'}\left(-\frac{\nabla^2_r}{2m}\right)e^{i\vec{k}'\vec{r}}=\sum_{\vec{k}'} a_{\vec{k}'}\frac{k'^2}{2m}e^{i\vec{k}'\vec{r}}=\sum_{\vec{k}'} a_{\vec{k}'}\,2\varepsilon_{k'}e^{i\vec{k}'\vec{r}}$

where we have used

(42) $\frac{k'^2}{2m} = \frac{k'^2}{2\frac{m_e}{2}} = 2\left(\frac{k'^2}{2m_e}\right) = 2\varepsilon_{\vec k'}$; $\varepsilon_{\vec k'} = \frac{k'^2}{2m_e}$

Hence (40) becomes

(43) $\sum_{\vec k'} a_{\vec k'}(E_1 - 2\varepsilon_{\vec k'})\, e^{i\vec k'\cdot\vec r} = \sum_{\vec k'} a_{\vec k'}\, V(\vec r)\, e^{i\vec k'\cdot\vec r}$; V_0 = volume

Now we multiply both sizes by $e^{-i\vec k\cdot\vec r}$ and integrate $\frac{1}{V_0}\int d\vec r$, so obtain

(44) $\sum_{\vec k'} a_{\vec k'}(E_1 - 2\varepsilon_{\vec k'})\underbrace{\left[\frac{1}{V_0}\int d\vec r\, e^{i\vec r(\vec k'-\vec k)}\right]}_{\delta_{\vec k,\vec k'}} = \sum_{\vec k'} a_{\vec k'}\underbrace{\left[\frac{1}{V_0}\int d\vec r\left(e^{-i\vec k\vec r}\, V(\vec r)\, e^{+i\vec k'\cdot\vec r}\right)\right]}_{V_{\vec k,\vec k'}}$

we obtain:

(45) $(E_1 - 2\varepsilon_{\vec k})\, a_{\vec k} = \sum_{\vec k'} a_{\vec k'}\, V_{\vec k,\vec k'}$

choosing now $V_{\vec k,\vec k'} = \lambda w_{\vec k} w^*_{\vec k'}$ one obtains

(46) $(E_1 - 2\varepsilon_{\vec k})\, a_{\vec k} = \lambda w_{\vec k}\underbrace{\sum_{\vec k'} w^*_{\vec k'} a_{\vec k'}}_{C} \Rightarrow (E_1 - 2\varepsilon_{\vec k})\, a_{\vec k} = \lambda w_{\vec k}\cdot C$

Hence

(47) $a_{\vec k} = \frac{\lambda w_{\vec k}\cdot C}{E_1 - 2\varepsilon_{\vec k}}$ which introduced in C gives

(48) $C = \lambda C \sum_{\vec k'} \frac{w_{\vec k'} w^*_{\vec k'}}{E_1 - 2\varepsilon_{\vec k'}} \Rightarrow \frac{1}{\lambda} = \sum_{\vec k'} \frac{|w_{\vec k'}|^2}{E_1 - 2\varepsilon_{\vec k'}} = f(E_1)$; (is a function of E_1)

Now we solve the second relation from (48) graphically. First we represent $f(E_1)$ in function of E_1 (see Fig 16), then considering attractive interaction ($\lambda<0$), we plot $\frac{1}{\lambda}$ in the figure 16 (line AB) and one looks for the intersection points. For $E_1>0$ (free states) we find several intersection (solution) points which are belong to the quasi-continuum of free states. However there is one solution at $E_1<0$ (bound state), the bonding energy (gap) being Δ (This energy is necessary to move the solution in the quasicontinuum of the free states. Note that the bound state has the lowest energy, so the Cooper pair indeed can be formed in a stable form. (see the calculations in L.N. Cooper: PR **104**, 1189 (1956)).

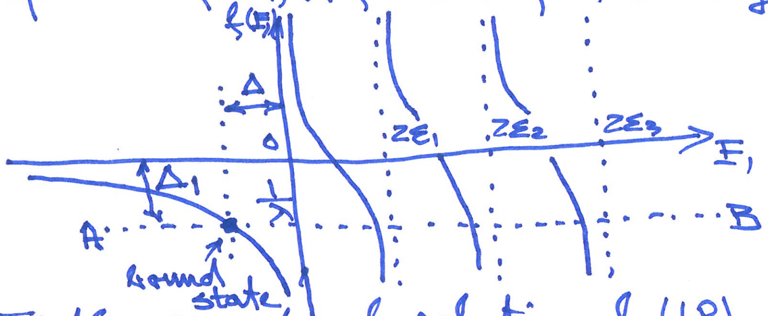

Fig 16: Graphical solution of (48).

In calculating the gap, we must suppose that the interaction is effective only in a thin layer around ε_F, of thicknes 2Ω. In this case, in mean-field, we obtain,

$$\Delta = 2\Omega \exp\left[-\frac{1}{N(\varepsilon_F)|\lambda|}\right] \qquad (49)$$

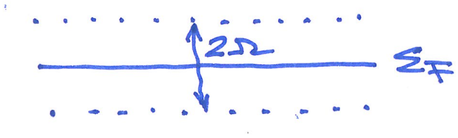

Fig 17: Region around ε_F where the interaction is effective

where $N(\varepsilon_F)$ is the density of states at Fermi level. As seen it is no threshold in the interaction strength for the emergence of the superconducting phase. It appears also in $|\lambda|\to 0$ limit.

For (49) see (A7) from the Appendix.

It also can be demonstrated that $\Psi(r) \sim \frac{1}{r^2}$, and the binding energy (Δ) decreases if $\vec{K} = \vec{K}_1 + \vec{K}_2 \neq 0$. Furthermore, calculating $\langle r^2 \rangle^{1/2} \sim \xi_0 \sim 10^{-4}$ cm, so the components of the Cooper pair are relatively far spaced each from another. (see (A25) in the appendix).

Observations:

1° $q \neq 0$ dependent Cooper pairs emerge when a coexistence between SDW (or CDW) and superconductivity occurs. (see PR B39, 12352 (1989); $q = Q$ holds).

2° In layered compound coexistence of in-plane and inter-plane superconducting gaps occur stabilizing the superconducting phase and increasing its critical temperature (see e.g. PR B37, 2247 (1988)).

3° The pair wave function $\Psi_2(\vec{r})$ in (25,26) can be expressed also in terms of sperical harmonics together with the interaction term $V(\vec{r})$ from (40). After this step different orbital momentum components in $\Psi_2(\vec{r})$ with fixed $l = 0, 1, 2, \ldots$ value appear, which are orthonormalized, hence not interfere. From these, from the properties of the spherical harmonics we know that, the $l = 2n+1$ (odd) $\vec{r}$ dependent wave functions change sign at $\vec{r} \to -\vec{r}$, so there are antisymmetric at the permutation of the two components of the Cooper pair, so instead of (36), the relation. In ^{3}He it happens that $l = 1$ case appears because the spin-fluctuations mediate the interaction not phonons

(50) $$\phi_l(r_1, r_2) = -\phi_l(r_2, r_1)\,; \quad l = 2n+1$$

(see A. J. Leggett: Rev. Mod. Phys. 47, 331 (1975)). But in this situation, instead of (38), one has

(51) $$\Psi_2(\vec{r}) = \phi^a_{(\vec{r})} \eta^s_1(\sigma_1, \sigma_2)\,; \quad \vec{r} = \vec{r}_1 - \vec{r}_2\,,$$

and as shown in (29), in this case parallel orientation of the spins in the Cooper pair appears, and the pair spin state $S=1$ occurs, which is a triplet state.

For Cooper pairs in a spin-triplet state the action of the external magnetic field is not accentuately dangereus. Consequently a such type of state is possible to appear in neutron stars, where the magnetic field of the star not destroys the condensed phase (See: Lecture Notes in Physics 578, 30 (2001) present as well in arXiv: astro-ph/0012209)

4° The superconductivity is called "conventional" if it can be described by BCS theory (PR 108, 1175 (1957)), which is a mean-field type of theory valid usually for small values of the inter-particle interactions. If this is not possible, we have "unconventional" superconductivity (Nature 555 (no: 7695), 151 (2018)). The high T_c materials ($T_c > 77$ K, the boiling point of N at normal pressure) are all unconventional superconductors.

5° Superconductors when appear introduce a gap in the energy spectrum

(52) $$\varepsilon = \sqrt{\varepsilon_K^2 + \Delta^2}\,; \quad \varepsilon_K = \frac{K^2}{2m}\,;$$ at the Fermi energy

Type I and Type II Superconductors:

The "Type" of the superconductor is given by its response to an external magnetic field.

Type I. superconductors: They have a single critical field H_{c1} above which the superconductivity is completely destroyed, and below which the external magnetic field is completely expelled from the superconductor (Meissner effect). Most pure elemental superconductors (except Nb (niobium), V (vanadium), Tc (technetium), and carbon nanotubes) are Type I superconductors, in most cases having $l=0$ (s-wave) Cooper pairs in spin singlet state.

Type II. superconductors: Almost all impure and compound superconductors are of this type. In this superconductors, for $H > H_{c1}$, but $H < H_{c2}$ a "mixed" state appears: magnetic flux penetrates in some points inside the superconductor, but remains no resistance to the flow of electric current as long as the current is not too large ($j < j_c$). The mixed state is caused by vortices in the electronic superfluid (called "fluxons"). The flux carried by these vortices are quantized. In vortices, the supercurrent circulates around the normal core of the vortex. The core (non-superconducting) has a size proportional to the coherence length (ξ). The supercurrents decay on a distance λ (= London penetration length. Type II superconductors appear when the Ginsburg-Landau κ parameter κ (see (53)) has the property $\kappa > \frac{1}{\sqrt{2}}$. Each vortex carries the flux quantum $\Phi_0 = \frac{h}{2e}$ and the vortices together build up a triangular lattice. When defects (impurities, dislocations) are present, this lattice is distorted. The phase diagram of these superconductors in function of H field is shown in Fig 18.

$$\kappa = \frac{\lambda}{\xi} \quad (53)$$

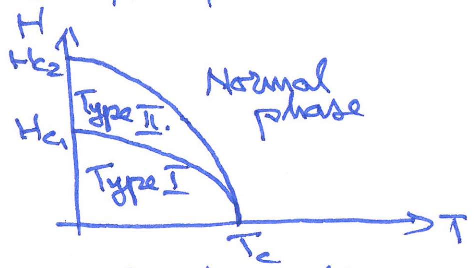

Fig 18: Phase diagram of Type II superconductors

Superinsulator

(see Nature 452, 613 (2008)). Cooper pairs are present but given by the disorder, they avoid each other, forming a state with infinitely large resistivity. Was observed e.g. in titanium nitride films: it is destroyed by strong critical magnetic field, and breaks down at a given critical current.

The Higgs mechanism:

First described by P.W. Anderson in PR 130, 439 (1963). When a continuous symmetry is broken, Goldston bosons appear. This are massless bosons e.g. phonons (acoustical) when the lattice is formed (continuous translational symmetry breaking) or magnons when the ferromagnetic state is formed (continuous rotational symmetry breaking). Massless in condensed matter language means no gap at $k \to 0$ (see Fig 19)

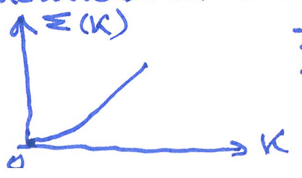

Fig 19: Goldstone boson behavior: $\lim_{k \to 0} E(k) = 0$ providing massless bosons.

I note that for particle physicists the "gap" is "rest mass" m_0 because the relativistic energy is $\varepsilon = \sqrt{(\hbar k c)^2 + m_0^2 c^4}$, where m_0 = rest mass. So for them

(54) $\varepsilon = \sqrt{\varepsilon_k^2 + \Delta^2} \to \varepsilon_k = pc = \hbar k c;\ \Delta = m_0 c^2$, so

(55) $\lim\limits_{k \to 0} \varepsilon_k = \begin{cases} 0, & \text{no rest mass} \\ \neq 0, & \text{rest mass} \end{cases}$

The Higgs mechanism says: 1) when together with a continuous symmetry also the gauge symmetry is broken, instead of the Goldston bosons Higgs bosons appear ("Higgs bosons eat Goldston boson") and these have a mass. When superconductivity appears a continuous gauge symmetry is broken, so the Cooper pair is a Higgs boson, and indeed as seen in (52) it has "rest mass" (i.e. introduces gap).

2.) Gauge bosons interacting with Higgs bosons gain mass. Gauge bosons are the bosons mediating the interactions in gauge fields. E.g. the photons ($\varepsilon = cp = c\hbar k$) are the gauge bosons (massless bosons) of the electromagnetic field. The photons in interaction with superconductivity (Meissner effect) indeed gain mass (rest mass) which can be measured (see Jackson, Classical Electrodynamics Section 12.8 (Proca equation), Sect. 12.9 (Photon mass in superconductor). The rest mass of photon is around $1Ry = 13.6eV$ (this is $m_0 c^2$ value.)

It is interesting to analyze how the photon rest mass fits with the Maxwell equations. The Maxwell equations are gauge symmetric since allow $\vec{A} \to \vec{A} + \mathrm{grad} f;\ V \to V - \frac{\partial f}{\partial t}$ for the vector and scalar potentials, so they describe photons with zero rest mass. However the Maxwell equations can be supplemented with a "mass term" (which means that in the Amper's law: $\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$, a new term $\nabla \times \vec{B} \sim -\mu_\delta^2 \vec{A}$ should appear where the photon rest mass $m_0 = \frac{\hbar}{c}\mu_\delta$, and $\vec{A}$ is the vector potential, i.e $\vec{B} = \nabla \times \vec{A}$). In this case the Maxwell equations become to be the Proca equations, see Sect. 12.8 from the Jackson's book). This term is provided by the London equation for superconductors, namely $\vec{J} = -\frac{q^2 n_q}{m_q}\vec{A}$, when q, n_q, m_q are the carrier charge, density and mass. Indeed in the presence of the electromagnetic field (photons have momentum) (see (56)). But for Cooper pairs with $\vec{k}, -\vec{k}$ electrons $\vec{p} = \vec{k}_1 + \vec{k}_2 = \vec{k} - \vec{k} = 0$, hence on obtains (57). But because the current density $\vec{J} = q n_q \vec{v}$, from (57) we obtain (58) which is the London equation for superconductors. This introduced in the Amper's law at time independent $\vec{E}$, so $\partial \vec{E}/\partial t = 0$ provides the desired (59), where μ_0 is the permeability, and $K = 1/\lambda$ where λ = the penetration depth of the superconductor. In order to see this we use $\vec{B} = \nabla \times \vec{A}$ and $\nabla \cdot \vec{A} = 0$ (Coulomb gauge) in (59) we find (60). The solution of (60) at a distance $\vec{r}$ from a surface in between a medium and superconductor (see Fig 20) inside the superconductor is

(56) $\vec{p} = m_q \vec{v} + q\vec{A}$

(57) $\vec{v} = -\frac{q}{m_q}\vec{A}$

(58) $\vec{J} = -\frac{q^2 n_q}{m_q}\vec{A}$

(59) $\nabla \times \vec{B} = -K^2 \vec{A}$, $\quad K^2 = \mu_0 \frac{q^2 n_q}{m_q}$

(60) $\nabla^2 \vec{A} - K^2 \vec{A} = 0$

(61) $\vec{A} = \mathrm{const}\,\vec{u}\, e^{-\frac{r}{\lambda}};\ \lambda = \sqrt{\frac{m_q}{\mu_0 n_q q^2}} \Rightarrow m_0 = \frac{\hbar}{c}\frac{1}{\lambda}$

So the photon rest mass can be directly measured by the penetration length λ, (=London penetration depth).

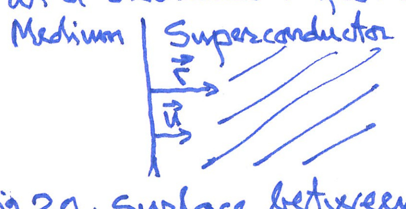

Fig 20: Surface between medium and superconductor

High T_c materials :

The property of superconductors to conduct electricity without loss of energy concentrated a huge research energy concentration in order to increase the critical temperature close to the room temperature. This history started in 1911 (Kamerling Ones) with $T_c = 4.2K$ for Hg (mercury). Then in 1913 $T_c = 7K$ for Pb (lead) and 1941 $T_c = 16K$ for NbN (niobium nitride). So 30 years were necessary for $\Delta T_c = 11.8K$. After further 46 years in 1987 Bednorz and Müller discovered $T_c = 35K$ in La based Cu perovskites, and the type of the layered material being known, again in 1987 in YBCO type of materials $T_c = 92K$ was found (PRL 58, 908 (1987)). In 1993 the up to date highest T_c at normal pressure has been measured: $T_c = 138K$ in $HgBa_2Ca_2Cu_3O_{8+\delta}$ (Nature 363, 56 (1993)). Meantime systems not containing CuO planes have been found superconductors at relatively high temperatures: oxypnictides: $(La\text{-}Sm)O_{1-x}Fe_x FeAs$ at $T_c = 55K$ (Europhys. Letters 83, 17002 (2008)).

Today, the highest measured T_c is $T_c = 203K$ for H_2S in 2015 but at extremely high pressure: 150 GPa (Nature 525, 73 (2015)). Interestingly this is conventional BCS, e.g. it has isotope effect.

Superconductors with finite-range potentials :

PRL 122, 157001 (2019) : Instead of $V_{kk'} = \lambda w_k w_{k'}$, se below (45) which means contact interaction, one has a finite range potential. $T_c(n)$ exhibits a maximum ("dome"), the superconductivity being of BCS type. So the presence of dome not necessarily means high T_c.

Ferromagnetic superconductors : PRL 122, 117002 (2019)

Nonuniform magnetic states can coexist with superconductivity. E.g. antiferromagnetic RMo_6S_8, RRh_4B_4 where the net magnetic moment at the scale of the superconducting coherence length ξ is zero. Similar effect appears here in P-doped $EuFe_2As_2$.

Appendix:

The gap value:

From (48) we start with the relation (see also Fig 15)

(A1) $\frac{1}{\lambda} = \sum_{k} \frac{|W_k|^2}{E_1 - 2\varepsilon_k}$, where for W_k we consider $W_k = 1$ if $|\varepsilon_k| \in [0, \Omega]$ and $W_k = 0$ if $|\varepsilon_k| > \Omega$. Furthermore, for Fig 16 we take $E_1 = -\Delta$ (where Δ is the gap value).

Hence from (A1) one has for the gap equation:

(A2) $-\frac{1}{\lambda} = \sum_{\vec{k}} \frac{1}{\Delta + 2\varepsilon_{\vec{k}}}$; But $\vec{k}$ runs on a symmetrical domain and the expression under the sum is even ($\varepsilon_k = \frac{k^2}{2m}$), hence $\sum_{k=-\Lambda}^{\Lambda} \to 2\sum_{k=0}^{\Lambda}$ where Λ is the cut-off in the momentum corresponding to $\varepsilon_k = \Omega$.

Hence, taking $\varepsilon_F = 0$ and

(A3) $\underbrace{\sum_{k=0}^{\Lambda}}_{k > k_F} \to \int_0^{\Omega} N(\varepsilon)d\varepsilon \to N(\varepsilon_F)\int_0^{\Omega} d\varepsilon$; where $N(\varepsilon_F)$ is the density of states at the Fermi energy, one obtains

(A4) $-\frac{1}{\lambda} = 2N(\varepsilon_F)\int_0^{\Omega} \frac{d\varepsilon}{\Delta + 2\varepsilon}$; We consider now $\lambda < 0$ (attractive). So $-\lambda = |\lambda|$. Then a variable change is effectuated: $x = 2\varepsilon \Rightarrow d\varepsilon = \frac{1}{2}dx$:

(A5) $\frac{1}{|\lambda|} = N(\varepsilon_F)\int_0^{2\Omega} \frac{dx}{\Delta + x} = N(\varepsilon_F)\ln(\Delta + x)\Big|_0^{2\Omega} = N(\varepsilon_F)\ln\frac{\Delta + 2\Omega}{\Delta}$, hence

(A6) $\exp\left[\frac{1}{|\lambda|N(\varepsilon_F)}\right] = \frac{\Delta + 2\Omega}{\Delta} \Rightarrow \Delta = \frac{2\Omega}{\exp\left[\frac{1}{|\lambda|N(\varepsilon_F)}\right] - 1}$; usually $|\lambda|N(\varepsilon_F) \ll 1$ hence

(A7) $\Delta = 2\Omega \exp\left[-\frac{1}{|\lambda|N(\varepsilon_F)}\right]$

Cooper pair with non-zero pair momentum:

In this case instead of $2\varepsilon_k$ in (A1) one has $(\varepsilon_{k_1} + \varepsilon_{k_2})$ where $(q > 0)$: $k_1 = k + \frac{q}{2}$; $k_2 = k - \frac{q}{2}$, and $k_1 - k_2 = q$ = the pair momentum. Hence instead of (A1) we have

(A8) $\frac{1}{\lambda} = \sum_k \frac{1}{E_1 - (\varepsilon_{k+\frac{q}{2}} + \varepsilon_{k-\frac{q}{2}})}$; Now if $k - \frac{q}{2} = k'$; $\Rightarrow k + \frac{q}{2} = k' + q$ one has from (A8) that ($E_1 = -\Delta$, and $-\lambda = |\lambda|$):

(A9) $\frac{1}{|\lambda|} = \sum_{k'} \frac{1}{\Delta + (\varepsilon_{k'} + \varepsilon_{k'+q})}$; But, for small q one has ($\frac{k'}{m} = v_F$)

$$\varepsilon_{k'+q} = \frac{(k'+q)^2}{2m} = \frac{k'^2}{2m} + \frac{2k'q}{2m} + \frac{q^2}{2m} =$$

$$= \varepsilon_{k'} + \frac{k'}{m}q + O(q^2) \cong \varepsilon_{k'} + v_F q$$

consequently:

(A10): $\frac{1}{|\lambda|} = \sum_{k'} \frac{1}{(\Delta + q v_F) + 2\varepsilon_{k'}}$; This is exactly (A2) but with $\Delta' = \Delta + q v_F$. Hence the result from (A6) automatically holds for Δ', and as a result, for $q \neq 0$:

(A11): $\Delta = \frac{2\Omega}{\exp\left[\frac{1}{|\lambda|N(\varepsilon_F)}\right] - 1} - v_F q$; Consequently for $K = q$ center of mass momentum of the pair, the binding energy (Δ) decreases by $v_F K$.

The pair wave function in r-space:

In Eq. (45), for $V_{\vec{k}\vec{k}'}$ one has in extending in spherical harmonics

(A12) $$V_{\vec{k}\vec{k}'} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} V_l(|\vec{k}|,|\vec{k}'|)\, Y_l^m(\Omega_{\vec{k}})\, Y_l^{m*}(\Omega_{\vec{k}'}) .$$

After this step one uses $a_{\vec{k}} = a(|k|,l)\, Y_l^m(\Omega_k)$, and in (45), since only $\sum_{k'}$ is present on the right side we obtain (decomposing in different l contributions that not interfere, so $\sum_l$ is not introduced)

(A13) $$(E_1 - 2\varepsilon_{\vec{k}})\, a(|k|,l) = \sum_{k'} V_l(|k|,|k'|) \underbrace{\sum_{m=-l}^{l} Y_l^{m*}(\Omega_{\vec{k}'})\, Y_l^m(\Omega_{\vec{k}'})}_{\frac{2l+1}{4\pi}\text{; (Unsöld's theorem)}} a(|k'|,l)$$

Taking

(A14) $\frac{2l+1}{4\pi} V_l(|k|,|k'|) = \lambda_l w_k^l w_{k'}^{l*}$ and, since we calculate at fixed l, we drop this index, we find (46) for fixed l

Hence at fixed l we find (47) (where $E_1 = -\Delta$)

(A15) $a_k = \dfrac{\lambda w_k C}{-\Delta - 2\varepsilon_k}$; $C = \sum_{k'} w_{k'}^* a_{k'}$ = a number, hence constant; $-\lambda = |\lambda|$ and $w_k = 1$, see at (A1):

Now from (37) one obtains

(A16) $$\phi^s(\vec{r}) = \sum_{\vec{k}} a_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} = |\lambda| C \sum_{\vec{k}} \frac{e^{i\vec{k}\cdot\vec{r}}}{\Delta + 2\varepsilon_{\vec{k}}};$$

In spherical coordinates, taking θ in between $\vec{r}$ and $\vec{k}$, we find

(A17) $$\phi^s(\vec{r}) = \frac{|\lambda| C}{(2\pi)^3} \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \int_{k_F}^{\Lambda} k^2 dk\, \frac{e^{ikr\cos\theta}}{\Delta + 2\varepsilon_k} \sin\theta;$$

since $dk_x dk_y dk_z = \sin\theta\, d\theta\, d\varphi\, k^2 dk$

Λ = cut-off in k

Now because

(A.18) $$\int_0^{\pi} \sin\theta\, e^{a\cos\theta} d\theta = \int_{-1}^{1} e^{ax} dx = \frac{1}{a}(e^a - e^{-a});$$ $a = ikr$ in our case

Furthermore $\int_0^{2\pi} d\varphi = 2\pi$, one obtains

(A.19) $$\phi^s(\vec{r}) = \frac{|\lambda| C}{(2\pi)^2} \frac{1}{ir} \int_{k_F}^{\Lambda} \frac{k\,dk}{\Delta + 2\varepsilon_k} (e^{ikr} - e^{-ikr});$$ since $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ $\Rightarrow$

(A.20): $$\phi^s(\vec{r}) = \frac{|\lambda| C}{2\pi^2} \frac{1}{r} \int_{k_F}^{\Lambda} \frac{k\,dk}{\Delta + 2\varepsilon_k} \sin kr = \frac{|\lambda| C}{2\pi^2} \int_{k_F}^{\Lambda} \frac{k^2 dk}{\Delta + 2\varepsilon_k} \cdot \frac{\sin kr}{kr}$$

(A.21) But $\int k^2 dk = \frac{1}{4\pi} \int d^3k \to \frac{1}{4\pi} N(0) \int d\varepsilon$; $kr = r\sqrt{2m\varepsilon}$; (since $\varepsilon = \frac{k^2}{2m}$)

consequently:

(A.22) $$\phi^s(\vec{r}) = \frac{|\lambda| C}{8\pi^3} N(0) \int_0^{\Omega} \frac{d\varepsilon}{\Delta + 2\varepsilon} \frac{\sin(r\sqrt{2m\varepsilon})}{r\sqrt{2m\varepsilon}};$$

Now we introduce a new variable as $r\sqrt{\varepsilon} = \sqrt{x}$ $\Rightarrow$ $d\varepsilon = \frac{dx}{r^2}$; $\varepsilon = \frac{x}{r^2}$

(A.23) $$\phi^s(\vec{r}) = \frac{|\lambda| N(0) C}{8\pi^3} \cdot \frac{1}{r^2} \int_0^{\Omega r^2} \frac{dx}{\Delta + \frac{2x}{r^2}} \frac{\sin[\sqrt{2mx}]}{\sqrt{2mx}};$$

For high r, $\Omega r^2 \to \infty$, so $a = 2m$; $\sqrt{ax} = y$; $ax = y^2$

$dx = \frac{1}{2m} y\,dy$

(A.24) $$\phi^s(\vec{r}) = \frac{|\lambda| N(0) C}{8\pi^3 r^2} \int_0^{\infty} \frac{\sin y}{y} \cdot \frac{y\,dy}{2m} \cdot \frac{1}{\Delta + \frac{2y^2}{2mr^2}};$$ so

(A.25): $$\phi^s(\vec{r}) = \frac{|\lambda| N(0) C}{16 m \pi^3} \cdot \frac{1}{r^2} \int_0^{\infty} \frac{\sin y\, dy}{\Delta + \frac{y^2}{mr^2}}; \Rightarrow \phi^s(\vec{r}) \sim \frac{1}{r^2}.$$