Lecture 11: The Renormalization Group:

From the Kadanoff block construction we have seen that close to T_c the starting Hamiltonian written in terms of the starting dynamical variables defined on sites can be substituted by a perfectly equivalent Hamiltonian written with block variables and the described physics will be perfectly identical. This is because the main property of the qualitative changes in many-body systems is the scale independence. Based on this idea Kadanoff succeeded to demonstrate theoretically the scaling laws that can be deduced only in this manner. The scale invariance also lead to the development of the theory of phase transitions leading to the Renormalization Group (Kenneth Wilson obtained the Nobel prize for it.). The basic idea is to transform the initial system defined on sites with sites variables in a system with blocks described by block variables. In the spin-system cases

$$\vec{S}_i \to \vec{S}_I \qquad (700)$$

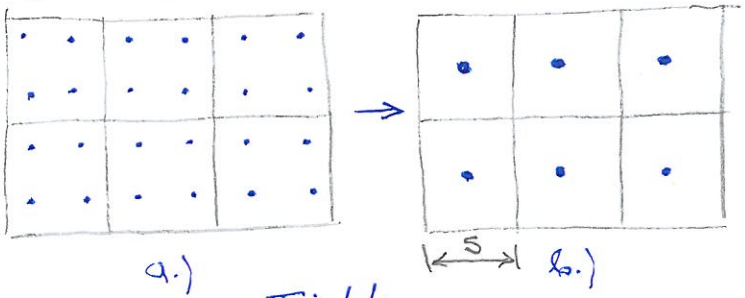


Fig 44

How a such transformation must be given?

1° It must not change the symmetries of the system, and

2° It must produce the same type of dinamical variable as was the starting one (so it is not a simple sum $\vec{S}_I \Rightarrow \sum_{i\in I} \vec{S}_i$; but it is a renormalised sum (see (189) from Lecture 5.):

$$\vec{S}_I = \frac{1}{L^x} \sum_{i\in I} \vec{S}_i \qquad (701)$$

The obtained new Hamiltonian with block variables should describe the same physics as the starting Hamiltonian defined with the starting variables defined on sites. Consequently, what an RG transformation does is a transformation

$$H' = K_s H \qquad (702)$$

where s is the block size. In fact, during a such type of transformation, also the thermodynamic potential changes, but close to T_c it should describe the same physics. (ϕ is constructed from the partition function Z, and Z from the Hamiltonian, so together with H also ϕ has to change.).

$$\phi' = K_s \phi \qquad (703)$$

Now: how big is the block size s? It do not matters, because the system is scale invariant. Even more, we can effectuate the transformation from Fig 44, and obtaining b.) we can consider it again as a starting lattice, and we can transform it again in a block Hamiltonian; Furthermore, this procedure can be repeated infinite many times. So the (702) transformation can be repeated and repeated again, hence can be written also in the form:

$$H_{n+1} = K_s H_n \qquad (704)$$

which means that startin from the Hamiltonian H_n obtained after the nth transformation, by applying the RG transformation K_s, we obtain the (n+1)th Hamiltonian. Since the physics remains the same in the critical region, it means that somehow, phase transformations will be

connected to the fixpoints of the RG transformation (702) or (704). Fixpoint it means that point of the parameter space where (702) satisfies $H'=H=H^*$, or in (704) $H_{n+1}=H_n=H^*$, where H^* is the fixpoint Hamiltonian satisfying

(705) $H^*=K_S H^*$; What is here the parameter space? The parameter space here in the space built up from the coupling constants of the Hamiltonian. Let us denote the vector built up from the coupling constants a_i of the Hamiltonian by

(706) $V=\{a_1,a_2,a_3,\dots a_n\}$; a_i = coupling constants.

In this manner what (702) does is in fact

(707) $V'=K_S V$; or written itiratively as in (704), one has

(708) $V_{n+1}=K_S V_n$; The fixpoint here will mean:

(709) $V^*=K_S V^*$; where $V^*=\{a_1^*,a_2^*,\dots a_n^*\}$; ($a_i^*$ is the fixpoint value of the coupling constant a_i; it is not a complex conjugate.)

Consequently, what the RG transformation does, is that it moves in the parameter space of the V vectors the point which corresponds to a given Hamiltonian

Fig 45

where n means the number of iterations from (704), or in other words the extent of the block that one has. Indeed, if n increases, also the block extent increases. For example at $n=1$ the block size is s_1, and at $n=2$ the block size is s_2, than performing in one step the transformation from the starting spins (i.e. dynamical variable) to the block spins obtained at the step $n=2$, we must use the block size

(710) $S=s_1\cdot s_2$

What we see in Fig 45 is a "flow trajectory" So up to this moment what we see from RG theory in connection to phase transitions is that to have a phase transition, the flow trajectories should somehow flow in a given point called fixpoint.

I note that since (704) causes the flow in Fig 45, we call Eq. 704 also as "flow equation".

How K_S must be effectuated:

Now the question is how one effectuates the RG transformation (i.e. K_S) effectively? For this one directly uses a relation of the Probability theory, namely:

Let us have a probability density $P(q_1,q_2,\dots q_n)$ where $(q_1,q_2,\dots q_n)$ are independent random variables. Now let us apply a constrain on the system as $q=f(q_1,q_2,\dots q_\ell)$, $\ell<n$. Now we are interested to find the new probability density wich one has under the constraint $q=f$. According to the theory of probability:

(711) $$P(q;q_{\ell+1},q_{\ell+2},\dots q_n)=\int dq_1\dots\int dq_\ell P(q_1,\dots q_n)\,\delta(q-f(q_1,q_2,\dots q_\ell))$$

Using this simple relation one effectuates the K_S (i.e. RG) transformation as follows:

(in discrete case $\int\to\Sigma$; The sums together form the trace T_r.)

For us, when one calculates the partition function, the role of the probability density is played by

(712) $\mathcal{P} \to e^{-\beta H} = e^{-\mathcal{H}}$, where all coupling constants of H multiplied by β give $\mathcal{H}$. For example if

(713) $H = -J\sum_{<ij>} \vec{S}_i\vec{S}_j$, J is the coupling constant, for $\mathcal{H}$ one has $\mathcal{H} = -K\sum_{<ij>} \vec{S}_i\cdot\vec{S}_j$, $K=\beta J$ for the coupling constant in $\mathcal{H}$.

Now according to (711), if $\mathcal{H}(S_i)$ is the starting Hamiltonian, and $\mathcal{H}'(S'_i)$ is the Hamiltonian obtained after the RG transformation which now depends on the block spins $S_I = S'_i$, one has:

(714) $e^{-\mathcal{H}'(S'_i)} = Tr\left[\prod_I \delta(S_I - f_I(S_i))\right] e^{-\mathcal{H}(S_i)}$; Tr is made over all S_i variables.

Here, because of

(715) $S_I = f_I(S_i)$ defines the block variables, the $f_I(S_i)$ function defines the block spin S_I (or in general the dynamical variable).

$f_I(S_i) \Leftarrow \frac{1}{L^x}\sum_{i\in I}\vec{S}_i = S_I$

Example 1: 1D Ising case:

Let us have a simple 1D Ising model with periodic boundary conditions

(716) $H = -J\sum_{i=1}^{N} S_iS_{i+1}$; so $\mathcal{H}(S_i) = -K\sum_{i=1}^{N} S_iS_{i+1}$, $K=\beta J$, $S_i = \pm 1$

For simplicity let us take the block formed from 3 consecutive spins (Fig 46)

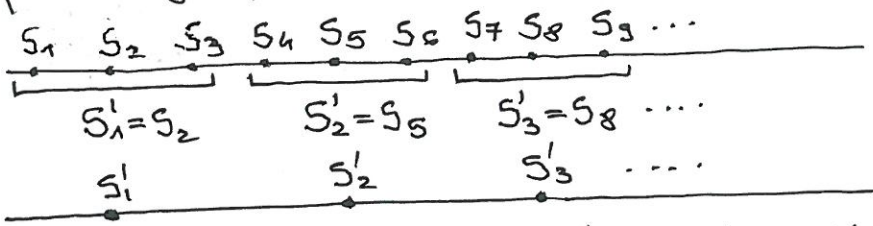


Fig 46

Such that, again for simplicity we take the block spin as being the middle spin of the block. This choise is in fact realistic at low temperatures where the middle spin usually will be aligned in the direction of the block spin. Furthermore, by this choise, the block spin S'_i will be exactly of the type as the starting spin S_i (see the requirement 2° below (700) on page 1, so will not be necessary for us to care about normalization (see (701)). Then, according to (714), one has:

(717) $e^{-\mathcal{H}'(S'_i)} = \sum_{S_1}\sum_{S_2}\cdots\sum_{S_N}\left[\delta_{S'_1,S_2}\,\delta_{S'_2,S_5}\cdot\delta_{S'_3,S_8}\cdots\right] e^{K\sum_i S_iS_{i+1}}$, where

In exponent:

(718) $K\sum_i S_iS_{i+1} = KS_1S_2 + KS_2S_3 + KS_3S_4 + KS_4S_5 + KS_5S_6 + KS_6S_7 + KS_7S_8 + KS_8S_9 + \ldots$

$= KS_1S'_1 + KS'_1S_3 + KS_3S_4 + KS_4S'_2 + KS'_2S_6 + KS_6S_7 + KS_7S'_3 + KS'_3S_9 + \ldots$

I_{12} (gives) I_{23} (gives) I_{34}

will analize together the terms denoted by $I_{i,i+1}$, where

(719) $\tilde{I}_{12} = \sum_{S_3}\sum_{S_4} e^{\underbrace{KS'_1S_3 + KS_3S_4 + KS_4S'_2}_{I_{12}}}$;

$\tilde{I}_{23} = \sum_{S_6}\sum_{S_7} e^{\underbrace{KS'_2S_6 + KS_6S_7 + KS_7S'_3}_{I_{23}}}$;

.... etc.

Note that $\sum_{S'_2}, \sum_{S_5}, \sum_{S_8}$ disappear from (717) because of δ function when one takes $S_2 = S'_1$, $S_5 = S'_2$, etc.

Hence we find from (717) that:

$$e^{-\mathcal{H}'(S_i')} = \tilde{I}_{12}\cdot\tilde{I}_{23}\cdot\tilde{I}_{34}\cdot\ldots\cdot\tilde{I}_{N1}\ ; \qquad (720)$$

Now let us calculate I_{12}: for this we use the relation (635) used in the study of the 2D Ising system in Lecture 10 (eq. (635)):

(721): $e^{KS_3S_4} = [\cosh K][1+S_3S_4\tanh K] = [\cosh K][1+S_3S_4 V]$; $(S_i=\pm 1)$;

$V=\tanh K$; $K=\beta J$; $\cosh K=\frac{e^K+e^{-K}}{2}$; $\sinh K=\frac{e^K-e^{-K}}{2}$; $\tanh K=\frac{\sinh K}{\cosh K}$

Then I_{12} from (719) becomes:

$$(722):\ \tilde{I}_{12} = \cosh^3 K \sum_{S_3=\pm1}\sum_{S_4=\pm1}(1+S_1'S_3V)(1+S_3S_4V)(1+S_4S_2'V)$$

Effectuating the product in the right side of (722) we obtain:

$$(723)\ (1+S_1'S_3V)(1+S_3S_4V)(1+S_4S_2'V) = (1+S_3S_4V+S_1'S_3V+S_1'S_3^2S_4V^2)(1+S_4S_2'V) =$$
$$= 1+S_3S_4V+S_1'S_3V+S_1'S_3^2S_4V^2+S_4S_2'V+S_3S_4^2S_2'V^2+S_1'S_3S_4S_2'V^2+$$
$$+S_1'S_3^2S_4^2S_2'V^3\ ;$$

Now under $\sum_{S_3}$, since $S_3=\pm1$, all terms with S_3 at odd power cancel out, and we find (for terms that remain a factor 2 appears since the sum has two term)

$$(724)\ \sum_{S_3}(1+S_1'S_3V)(1+S_3S_4V)(1+S_4S_2'V) = (1+S_1'S_4V^2+S_4S_2'V+S_1'S_2'V^3)\times 2$$

Now similarly, effectuating the sum over $\sum_{S_4}$ we obtain

$$(725)\ \sum_{S_3}\sum_{S_4}(1+S_1'S_3V)(1+S_3S_4V)(1+S_4S_2'V) = 4(1+S_1'S_2'V^3)$$

Consequently, from (722) one obtains:

$$(726)\ I_{12} = 4\cosh^3 K\,[1+S_1'S_2'V^3]\ ;$$

Now we denote

$$(727)\qquad \tanh K' = V^3 = \tanh^3 K\ ,$$ hence (726) becomes (see (721) as well)

$$(728)\ \tilde{I}_{12} = \left[\frac{4\cosh^3 K}{\cosh K'}\right]\cdot\left[\cosh K'(1+S_1'S_2'\tanh K')\right] = \left[\frac{4\cosh^3 K}{\cosh K'}\right]\cdot e^{K'S_1'S_2'} = g(K,K')e^{K'S_1'S_2'}$$

Similarly, all $I_{i,i+1}$ terms in (720) will give

$$(729)\ \tilde{I}_{i,i+1} = g(K,K')\,e^{K'S_i'\cdot S_{i+1}'}\ ,$$ hence reintroducing in (720) we find:

$$(730)\ e^{-\mathcal{H}'(S_i')} = g_{(K,K')}^{N'}\cdot e^{K'(S_1'S_2'+S_2'S_3'+S_4'S_5'+\ldots+S_{N'-1}'S_{N'}'+S_{N'}'S_1')}$$

Here $N'=\frac{N}{3}$ represents the number of S_i' spins (i.e. block spins).

Now because of (727), $K'=f(K)$; (i.e. $K'=\text{arctanh}[\tanh^3 K]$, in fact $g(K,K')=g(K)$, so is only K dependent, and one can write:

$$(731)\ g_{(K)}^{N'} = e^{\ln g_{(K)}^{N'}} = e^{N'\ln g(K)}\ ;$$

Hence (730) becomes

$$(732)\ e^{-\mathcal{H}'(S_i')} = e^{N'\ln g(K)+K'\sum_{i=1}^{N'}S_i'S_j'}\ ,$$ that is

$$(733)\qquad \mathcal{H}'(S_i') = -K'\sum_{i=1}^{N'}S_i'S_j' + \underbrace{(N'\ln g(K))}_{\text{constant, not depends on } S_i'}$$

Because a constant not influences the physics provided by a Hamiltonian, but only shifts the spectrum one has in fact from the starting point in (716):

(734) $\mathcal{H}(S_i) = -K\sum_i S_i S_{i+1}$, after the RG transformation

$\mathcal{H}'(S_i') = -K'\sum_i' S_i' S_{i+1}'$;

So as seen the Hamiltonian remains of the same form (so describes the same physics), only the coupling constant changes. So the equation (707) becomes in our case (see (727)):

(735) $\boxed{\tanh K' = \tanh^3 K}$; This is the equation $V' = K_S V$; where the vector $V = \{K\}$ has only 1 component.

The fixed points:

According to (709) the fixpoint are provided by

(736) $\tanh K^* = \tanh^3 K^*$; Denoting $x = \tanh K^*$, one has the equation: (x must be real):

(737) $x = x^3 \Rightarrow x_1^* = 0$; $x_2^* = 1$; these correspond to $K_1^* = 0$; $K_2^* = \infty$ so from the point of view of the temperature ($K = \frac{J}{k_B T}$), one has $T_1^* = \infty$; $T_2^* = 0$. Now in the language of the flowing equation:

(738) $x_{n+1} = x_n^3$, and because $x < 1 \Rightarrow x_{n+1} < x_n$, so the flow trajectory show displacement from $x_2^* = 1$ to $x_1^* = 0$ (see Fig 47)

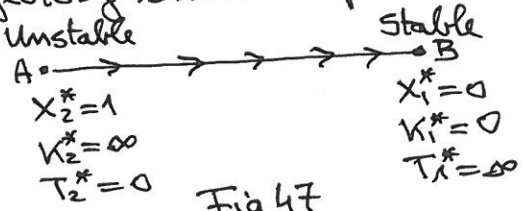


Fig 47

In Fig 47 two fixed point appear:

A: From here everything goes out. This is an unstable fixed point. It would correspond to the $T=0$ ordered phase in the 1D Ising model, but the corresponding fixpoint being unstable, that it means that an infinitesimal perturbation in the neighborhood of A moves the point which corresponds to the Hamiltonian of the system to the fixpoint B. One says that an unstable fixpoint (see for the general case Fig 48) "has no critical surface": i.e. there is no region in the parameter space from where it would be possible to enter there. If you are exactly in the unstable fixpoint you remain there (i.e: $x_n = 1 \Rightarrow x_{n+1} = 1$). But if you are outside you will move far away from there.

Fig 48: unstable fixpoint

Critical surface: it is not necessarily a 2D manifold: is a region around the fixpoint (not the whole volume, only a part of it) from where it is possible to flow inside the fixpoint. The unstable fixpoint (often called also: repulsive fixpoint) does not have critical surface, and not describe real phase transitions.

B. The fixpoint B is a stable fixpoint. In general it looks like Fig 49. It is also called "attractive" fixpoint. As seen in Fig 47 also (there it corresponds to $T=\infty$), it attracts everything. It corresponds to a necessity: in Fig 47, it means that at $T \to \infty$ you obligatory reach the completely disordered phase. So not means phase transition. By approaching this fixpoint, the coupling constant gets smaller and is 0 at B.

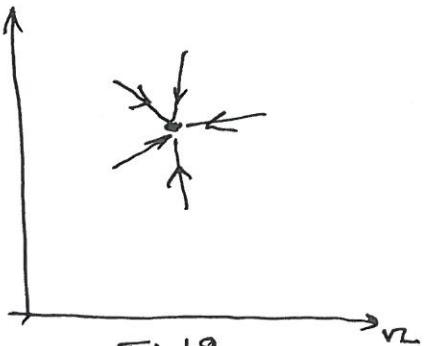


Fig 49
Stable fixpoint

So in order to have a phase transition, a fixpoint should be present at $T \neq 0$. Since here is absent, it means that ordered phase in 1D cannot appear for the studied system. But how should look like the fixpoint which means phase transition, if it is not like Fig 48, nor like Fig 49? It looks like Fig 50. It is called "critical fixpoint". It is not attractive, but has critical surface: the line CAB. The fixpoint is in A.

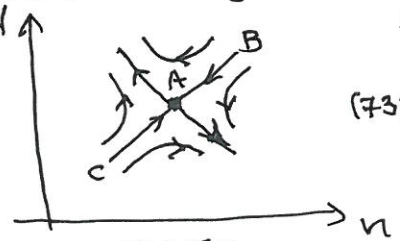


Fig 50
Critical fixpoint

(739): So: phase transitions are provided by critical fixpoints, the fixpoints that have critical surface

Let us judge the difference between Fig 49 and Fig 50 in the following way: For V in (706) we calculate the correlation length and we obtain a value, say $\xi(V)$; Then, we effectuate the RG transformation K_s: $V' = K_s V$ and in the new system, described by V', we again calculate the correlation length, obtaining $\xi(V')$. But during the transformation the length scale is reduced by $s > 1$, so one has

(738) $$\xi(V') = \frac{\xi(V)}{s}$$

But then, if we arrive to a fixed point, we obtain:

(739) $$\xi(V^*) = \frac{\xi(V^*)}{s};$$ which has two possible solutions, namely:

(740). a.) $\xi(V^*) = 0$;

b.) $\xi(V^*) = \infty$;

The first case a.) corresponds to Fig 49, and the second case b.) to Fig 50. And since for a phase transition providing qualitative change one needs $\xi \to \infty$, Fig 50 will correspond to a phase transition. But you can ask: even at a qualitative level, why is not vice versa? The reason is, that in a transition point you have the chance to choose in between 2 possibilities (i.e. 2 phases); But Fig 49 not offers different possibilities to chose in between.

The critical surface: Is the line CAB from where you surely enter in the critical fixpoint, (i.e. you surely has a phase transition). But that happens only for $T = T_c$. Consequently:

(740): The equation of the critical surface is the equation for the critical temperature ($T = T_c$) → or "critical manifold"

The critical surface is often called also „the basin of attraction" of the critical fixed point. Since on the critical surface $T = T_c$ (i.e. $\xi = \infty$) it results that all points in the basin of attraction have infinite correlation length.

In the same time a) from (740) is related to "trivial" fixed point (like Fig 49). These not describe phase transitions, but the bulk phases of the system.

Linearized Renormalization Group transformation:

Suppose one has the RG transformation $V'=K_s V$; If we are close to the fixed point then

(741) $V=V^*+\delta V$; $V'=V^*+\delta V'$; and in fact K_s connects now $\delta V'$ to δV. The transformation that indeed mathematically connects $\delta V'$ to δV is the "linearized" RG transformation, and will be denoted by R_s^e. But δV and $\delta V'$ which are linked by R_s^e in the forum

(742): $\delta V'=R_s^e \delta V$; can be considered to be vectors, which means that R_s^e is in fact a matrix.

Indeed: (743): $\delta V=\begin{pmatrix} V_1-V_1^* \\ V_2-V_2^* \\ \vdots \\ V_n-V_n^* \end{pmatrix}$; $\delta V'=\begin{pmatrix} V_1'-V_1^* \\ V_2'-V_2^* \\ \vdots \\ V_n'-V_n^* \end{pmatrix}$; where V_α there are the coupling constants before the transformation, and V_α' are the coupling constants after the transformation: $\alpha=1,2,\dots n$.

Then the matrix R_s^e has the matrix components

(744): $(R_s^e)_{\alpha\beta}=\left(\frac{\partial V_\alpha'}{\partial V_\beta}\right)\Big|_{V=V^*}$; and in fact (742) can now be written as (using the Einstein convention)

(745): $(V_\alpha'-V_\alpha^*)=(R_s^e)_{\alpha\beta}(V_\beta-V_\beta^*)$; or, taking into account also the iteration steps n, see (708), one has

(746): $(V_\alpha^{n+1}-V_\alpha^*)=(R_s^e)_{\alpha\beta}(V_\beta^n-V_\beta^*)$;

Based on (746), a successive application of the RG transformation m-times will give:

$$\delta V^{n+m}=(V^{n+m}-V^*)=(R_s^e)^m(V^n-V^*)=(R_s^e)^m\delta V^n ; \quad (747)$$

This shows that in fact, in order to reach the fixpoint we should consider

(748) $(R_s^e)^m\big|_{m\to\infty}$: introduce us in the fixpoint.

Eigenvectors, eigenvalues:

The matrix R_s^e has eigenvalue equation, so eigenvalues and eigenvectors:

(749) $(R_s^e)_{\alpha\beta}e_\beta^{(i)}=\lambda_{i(s)}e_\alpha^{(i)}$; where i labels the eigenvalue.

But, based on $s'=s_1\cdot s_2$ (see (710)), one has:

(750) $R_{s_1}^e R_{s_2}^e=R_{s'}^e \Rightarrow \lambda_i(s_1)\lambda_i(s_2)=\lambda_i(s')$, where $s'=s_1\cdot s_2$

But from (750): $\lambda_i(s)\lambda_i(s)=\lambda_i(s^2) \Rightarrow$ the eigenvalue should be a power function of s, that is

(751) $\lambda_i(s)=s^{y_i}$; where y_i is the exponent of the eigenvalue.

In the same time $e_\alpha^{(i)}$; $\alpha=1,2,\dots n$ form the column vector eigenvector

(752) $e^{(i)}=\begin{pmatrix} e_1^{(i)} \\ e_2^{(i)} \\ \vdots \\ e_n^{(i)} \end{pmatrix}$; which corresponds to the eigenvalue $\lambda_i(s)$.

From the other side, we know that the set of eigenvectors build ups a base for the linear space in which we work, which means that the vector δV can be expressed in function of this base:

(753): $\delta V = \sum_j t_j e^{(j)}$; where t_j are numerical coefficients.

Hence, using (742)

$$\delta V' = (R_s^e)\delta V = \sum_j t_j \underbrace{(R_s^e) e^{(j)}}_{\lambda_j(s) e^{(j)}\ ;\ \text{see (749)}\ ;} = \sum_j t_j \lambda_j(s) e^{(j)} = \sum_j t_j s^{y_j} e^{(j)} \quad ; \quad (754)$$

↓ we use (751)

Consequently:

(755) $\delta V' = \sum_j t'_j e^{(j)}$; $t'_j = t_j s^{y_j}$;

However $\delta V' = V' - V^*$, which then means that

(756) $V' = V^* + \sum_j t_j s^{y_j} e^{(j)}$;

Since $s > 1$, this means that ; if

(757)

1) $y^j > 0 \Rightarrow \lambda_j(s) > 1$; and s increases (so we effectuate RG transformation) we roll away (get off) from the fixed point.

2) $y^j < 0 \Rightarrow \lambda_j(s) < 1$; and s increases, we approach (come closer) to the fixpoint

3) $y^j = 0 \Rightarrow \lambda_j(s) = 1$; and s increases, our distance from the fixpoint remains the same.

The three cases above are given the following terminology:

1.) Relevant: eigenvalues, eigenvectors, directions: these get us away from the fixed point

2.) Irrelevant: eigenvalues, eigenvectors, directions: these approach us to the fixed point

3.) Marginal: eigenvalues, eigenvectors, directions: these not modify the distance to the fixpoint

The eigenvectors which are irrelevant span the critical manifold (= the critical surface).

These terms (relevant, irrelevant, marginal) are connected to a given fixed point. Particular terms of the Hamiltonian can be relevant for one fixed point and irrelevant for another fixed point.

We further observe that all λ_j eigenvalues can be put in order:

(758) $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \dots$

Global properties of the RG flows: these determine the phase diagram. All the points from the parameter space that during the RG successive transformation flows in the same fixed point, are in the same phase

Universality: different points of the parameter (coupling constant) space correspond to different Hamiltonians. If they flows to the same critical fixed point there are in the same universality class. So their critical behavior is the same.

Critical exponents of a phase transition.

Let us place ourselves in the neighborhood of a critical fixpoint (this corresponds to phase transitions). Let be in spontaneous condition where $h=0$, so the external field conjugated to the order parameter, (or if we do not have order parameter) all external fields are zero. Let be this fixpoint V^*. Now, according to (755) one has:

(759) $$V' = V^* + \sum_j t_j s^{y_j} e^{(j)} \; ;$$

Now we linearize the RG transformation around V^* as shown in (744), we hence obtain the $(R^l_s)_{\alpha\beta}$ matrix. We deduce its eigenvalues λ_j

(760) **If between the deduced eigenvalues is only 1 relevant eigenvalue, and the other eigenvalues are irrelevant, then the fixed point is a critical fixed point. ($h=0$)**

As we did with eigenvalues in (758), we can put in order also the exponents y_j of the eigenvalues ($\lambda_j(s) = s^{y_j}$). We obtain:

(761): $y_1 > y_2 > y_3 > \dots$

If one has only 1 relevant eigenvalue, see (757), it means:

(762): $y_1 > 0$; $y_2 < 0$, $y_3 < 0$, ...

Now let us turn back to (759) with (762). With successive application of the RG transformation, which via (748) can be interpreted as s increases, all $y_j < 0$, $j > 1$ contributions in (759) provide non-semnificative values, and one remains with

(763): $$V' = V^* + t_1 s^{y_1} e^{(1)}$$

Since $y_1 > 0$, the second term in (763) pushes us away from the critical point V^* at arbitrary T. The unique possibility to reach V^* is to place us on the critical surface: then the second term in (763) must disappear, that is $t_1 \sim |t|$, where $t = \frac{T - T_c}{T_c}$ is the reduced temperature. Then one has

(764): $t_1 = K|t|$; But since $\xi \sim |t|^{-\nu} \Rightarrow \xi \sim \frac{1}{|t|^\nu}$; $|t|^\nu \sim \frac{1}{\xi}$; $|t| \sim \frac{1}{\xi^{1/\nu}}$ so

(765): $t_1 \sim K \xi^{-\frac{1}{\nu}} \Rightarrow t_1 = K' \xi^{-\frac{1}{\nu}}$ where K' is a constant; Using (763):

(766): $$V' = V^* + t_1 K' \frac{s^{y_1}}{\xi^{1/\nu}} e^{(1)}$$; But each length (as s = block extent) for $T \to T_c$ appears in expressions only as (s/ξ), hence

(767): $\boxed{\frac{1}{\nu} = y_1}$; This is the way in which RG provides the critical exponent ν from the exponent of the unique relevant eigenvalue λ_1.

If one has ν, from the Josephson equality ($2 - \alpha = \nu d$) we obtain also the critical exponent α as

(768): $$\boxed{\alpha = 2 - \frac{d}{y_1}}$$

Let us now consider that also $h \neq 0$. In this case

(769) In between the eigenvalues of $(R_s^L)_{\kappa\beta}$ we must have two relevant eigenvalues, and the remaining eigenvalues must be all irrelevant in order to have a critical fixed point.

This it means that in the sequence of eigenvalue exponents in (761) now one has instead of (762) the situation:

(770). $y_1>0$; $y_2>0$; $y_3<0$, $y_4<0$, $y_5<0$, ...

Now in order to se mathematically what is happening, let us turn back to lectures 4 and 5, where one had:

$t' = L^y t$, $h' = L^x h$ (169); $a_t = \frac{y}{d}$, $a_h = \frac{x}{d}$ (172); $\alpha = 2 - \frac{1}{a_t}$; (224); $\beta = \frac{1-a_h}{a_t}$; (209);

The equation numbers are from the hand-written Lecture notes

L = blocksize.

Now we also have these relations, but with other notations, namely:

(771) $L \to s$; $y \to y_1$; $x \to y_2$; where y_1, y_2 there are the two relevant eigenvalue exponents in (770).

Then the transciption of the presented equations now become:

(772) $t' = s^{y_1} t$; $h' = s^{y_2} h$; $a_t = \frac{y_1}{d}$; $\alpha = 2 - \frac{d}{y_1}$; $a_h = \frac{y_2}{d}$; $\beta = \frac{1}{a_t} - \frac{a_h}{a_t} = \underbrace{2-\alpha}_{1/a_t} - \frac{y_2}{y_1}$; (usually y_2 is the higher exponent.)

So from the exponents of the 2 relevant eigenvalues we obtain α (already presented in (768)) and now also β as:

(773). $\boxed{\beta = 2 - \alpha - \frac{y_2}{y_1}}$

Now one has from RG three critical exponents: α, β, ν, and the rest can be deduced from the scalling laws:

(774)
- Griffiths: $\alpha + \beta(\delta+1) = 2 \Rightarrow \delta = \frac{2-\alpha}{\beta} - 1 = \frac{2-\alpha}{2-\alpha-\frac{y_2}{y_1}} - 1 = \frac{\frac{y_2}{y_1}}{2-\alpha-\frac{y_2}{y_1}}$
- Widom: $\gamma = \beta(\delta-1) = (2-\alpha-\frac{y_2}{y_1})(\frac{2-\alpha}{\beta} - 2)$
- Fisher: $(2-\eta)\nu = \gamma \Rightarrow \eta = 2 - \frac{\gamma}{\nu}$;

Example: 2D Ising model on a triangular lattice:

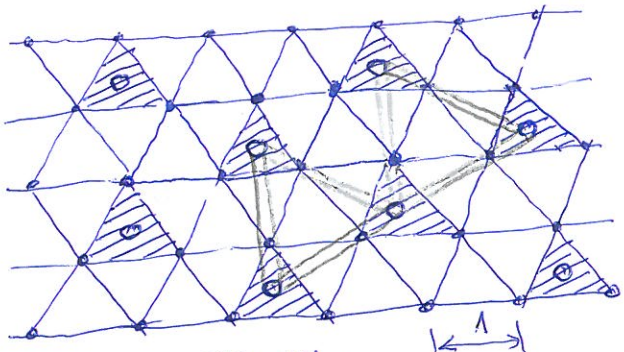
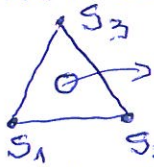


Fig 51:
2D triangular lattice; High triangles represent the new lattice. Shaded areas are the blocks;

The block spin is taken as



$\Rightarrow S_I = \text{Sign}[S_1+S_2+S_3]$; (775)

Block extent: $s = \sqrt{3}$

The Hamiltonian:

$$\mathcal{H} = -K \sum_{\langle ij \rangle} S_i S_j - H \sum_i S_i \; ; \quad (776)$$

$K = \frac{J}{k_B T}$; $H = \frac{h}{k_B T}$; $S_i = \pm 1$

Using the procedure we applied for 1D case, one obtains the RG equations become (after several approximations):

$$(777)\cdot\begin{cases} K' = 2K\Phi^2(K); \\ H' = 3\Phi(K)\cdot H; \end{cases} \quad \Phi(K) = \frac{e^{3K}+e^{-K}}{e^{3K}+3e^{-K}};$$

(See: Lectures on the phase transitions and the renormalization group; N. Goldenfeld Perseus Book, 1992, pg. 257) Section 9.6;

As you see this is a quite simple example in which the variables are decoupled. The fixpoint equations are

$$(778)\begin{cases} K^* = 2K^*\Phi^2(K^*); \\ H^* = 3\Phi(K^*)H^*; \end{cases}$$

We obtain $K_1^*=0, K_2^*=\infty$; $\Phi(K_3^*)=\frac{1}{\sqrt{2}}$ (gives $K_3^*=0.34...$)

To each of these is coupled ($h>0$):

$$H_1^*=0;\ H_2^*=\infty.$$

The critical fixed point is:

$$(779).\quad K_3^*=0.34...;\ H_1^*=0$$

We now linearize the RG transformation around the critical fixed point (779). We obtain according to (744), ($v=(K,H)$):

$$(780):\ R_S^e = \begin{bmatrix} \frac{\partial K'}{\partial K}\big|_{v=v^*} & \frac{\partial K'}{\partial H}\big|_{v=v^*} \\ \frac{\partial H'}{\partial K}\big|_{v=v^*} & \frac{\partial H'}{\partial v}\big|_{v=v^*} \end{bmatrix} = \begin{bmatrix} 2\Phi^2(K^*)+2K^*\Phi(K^*)\frac{\partial\Phi}{\partial K}\big|_{K=K^*} & 0 \\ 0 & 3\Phi(K^*) \end{bmatrix};$$

Since the matrix is diagonal, the eigenvalues are: $\left(\frac{\partial\Phi}{\partial K}=\frac{8e^{2K}}{e^{3K}+3e^{-K}}\right)$

$$(781):\begin{cases} \lambda_1 = S^{y_1} = 2\Phi^2(K^*)+2K^*\Phi(K^*)\frac{8e^{2K^*}}{e^{3K^*}+3e^{-K^*}}\big|_{K^*=0.34...} = 1.77... \\ \lambda_2 = S^{y_2} = 3\Phi(K^*)\big|_{K^*=0.34...} = 2.12... \\ S=\sqrt{3}=1.73... \end{cases}$$

consequently one obtains:

$$(782)\quad y_1=\frac{\ln\lambda_1}{\ln S}=\frac{\ln 1.77}{\ln 1.73}=1.042;\quad \nu=\frac{1}{y_1}=0.96;\ \alpha=0.08$$
$$y_2=\frac{\ln\lambda_2}{\ln S}=\frac{\ln 2.12}{\ln 1.73}=1.37\qquad \beta=0.605;$$

Calculated from the scaling laws, the other critical exponents become:

Onsagers exact value: $y_1=1$, hence $\nu=1, \alpha=0; y_2=\frac{15}{8}=1.875; \beta=\frac{1}{8}=0.125$

$$(783):\ \delta=2.17;\ \gamma=0.71;\ \eta=1.26$$

I note that these are not the exact values because strong approximations have been made in deducing (777). But the procedure exemplifies the technique used by R.G.

From K^*, the critical temperature becomes ($K=\frac{J}{k_BT}$)

$$(784)\quad k_BT_c = 0.34J;$$

The mean field value (see (377)) is $k_BT_c=\frac{S(S+1)}{3}Jz$ with $S=1, z=6$ gives

$$(785)\quad k_BT_c\big|_{\text{mean-field}}=4J;$$

so it is almost 12 times is higher. So clearly can seen that the mean-field overestimates the ordering tendencies.

$$(786)\quad k_BT_c\big|_{\text{Onsager}}=0.27J$$

↳ This is the exact result

I would like to present now much higher quality results deduced from $(\phi^2;\phi^4)$ theory (universality class of real 3D gases). $(\varepsilon = 4-d)$:

(787)
$$\alpha = \frac{\varepsilon}{6} + \mathcal{O}(\varepsilon^2) \qquad \alpha' = \frac{1}{6} + \mathcal{O}(\varepsilon^2)$$
$$\beta = \frac{1}{2} - \frac{\varepsilon}{6} + \mathcal{O}(\varepsilon^2) \qquad \beta = \frac{1}{3} + \mathcal{O}(\varepsilon^2)$$
$$\gamma = 1 + \frac{\varepsilon}{6} + \mathcal{O}(\varepsilon^2) \qquad \gamma = \frac{7}{6} + \mathcal{O}(\varepsilon^2)$$
$$\delta = 3 + \varepsilon + \mathcal{O}(\varepsilon^2) \qquad \delta = 4 + \mathcal{O}(\varepsilon^2)$$
$$\nu = \frac{1}{2} + \frac{\varepsilon}{12} + \mathcal{O}(\varepsilon^2) \qquad \nu = \frac{7}{12} + \mathcal{O}(\varepsilon^2)$$
$$\eta = 0 + \mathcal{O}(\varepsilon^2) \qquad \eta = 0 + \mathcal{O}(\varepsilon^2)$$

Provides in $d=3$

So the Guggenheim result $\beta = \frac{1}{3}$ is recovered.

Renormalization Group in $\vec{\kappa}$-space:

In $\vec{r}$-space we integrate out the inside of the block which after the RG transformation is no more present as an operator in H'. But small $\vec{r}$ corresponds to high $\vec{\kappa}$, so in $\vec{\kappa}$ space:

(788): During an RG transformation we integrate out the $|\vec{\kappa}|$ values in between Λ and Λ/s, where Λ is the cut-off.

For example if the dynamical variable is $\sigma_\alpha(\vec{r})$; (α is the vector component) one has

(789): $$\sigma_\alpha(\vec{r}) = \frac{1}{\sqrt{S^d}} \sum_{\kappa<\Lambda} \sigma_\alpha(\vec{\kappa}) e^{i\vec{\kappa}\cdot\vec{r}}$$; So we obtain for the RG transformation:

(790): $$e^{-\mathcal{H}'(\sigma')} = \int e^{-\mathcal{H}(\sigma)} \left[\prod_\alpha \prod_{\frac{\Lambda}{s}<\kappa<\Lambda} d\sigma_\alpha(\vec{\kappa})\right] \; ; \; \kappa = |\vec{\kappa}|$$

Since the resolution after an RG transformation decreases with the new cut-off will be now $\frac{\Lambda}{s}$. Since high $\vec{\kappa}$ means high energy, we practicaly during RG integrate out the high energy behaviour. Hence, on the flowing trajectories:

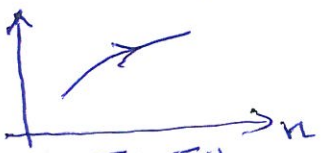


Fig 52

The direction indicates decrease of the $\vec{\kappa}$ values.

Consequently, the same universality class will mean the same "low energy behaviour". Low energy means here energies around the ground state.