

I. Introduction:IMPORTANCE

1° In order to understand the value in physics of the theory of phase transitions, I mention that if after several centuries the physicists ask themselves about what was most important in the development of physics in the twentieth of century, the answer to this question certainly will contain: quantum mechanics, theory of relativity and renormalization group. The present lecture will try to make you understand the last of this subjects. Indeed, here we will try to understand together the laws governing the qualitative changes in many-body systems, ending (inclusively) with the renormalization group. I only mention that at my home, the second semester of this lecture starts considering that the renormalization group, and vertex-function techniques are well understood and we try to demonstrate the presence of qualitative changes in many-body systems based on these 2 "heavy artilleries" of the theory of phase transitions. I am mentioning this in order to make you understand what will be the stage of our understanding in this field at the end of this lecture: we will arrive up to the understanding of the renormalization group. Parallel to this, the vertex-function (and Green-function) techniques are assimilated in a parallel lecture on "Many-body physics" (which practically means the application of field theory to statistical physics).

WHY IMPORTANT

2° In order to make you understand from where is coming the importance of this field, I ask another question: have you ever wondered about the role of the physicist in the society? If you do that, you will conclude that practically our role is to put the matter in a form which has some elevated (or desired) properties for a given application. But the transformation of properties of the matter needs usually qualitative changes. The theory of phase transitions and critical phenomena analyzes exactly the laws which drive qualitative changes in matter. That is why it has a central role in physics.

AN OBSERVATION:

3° What I have underlined till now is that here we are interested in qualitative changes in properties in many-body systems (evidently without changing the chemistry of the system - e.g. if water molecules were present, after the changes of the properties again water molecules will be present). Now, qualitative changes in properties emerge at $T \neq 0$ and $T = 0$ as well. These 2 situations are not completely equivalent.

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Qualitative changes at $T=0$ are analyzed in the "Theory of quantum phase transitions". Qualitative changes at $T \neq 0$ are analyzed by the "Theory of phase transitions". Since our lecture is on phase transitions, we will be interested to understand qualitative changes in properties at $T \neq 0$. However similarities are present, the $T=0$ and $T \neq 0$ cases are not equivalent. At $T=0$ the quantum fluctuations play the destructive role (i.e. work against the ordering tendency of the system), while at $T \neq 0$ the destructive role is played by the thermal fluctuations. I will present to you coherently the $T \neq 0$ case, but sometimes, at important points, I will be mentioning what is happening at $T=0$ in similar situations.

II. The phase notion and behavior of the thermodynamic potential

1. WHO DETERMINE THE STATE OF THE SYSTEM

At $T=0$, if a system is not perturbed from outside, it reaches the state with the minimum energy E . Consequently, at $T=0$ the ground state energy E is the quantity which fixes the state of the system.

Contrary to this, at $T \neq 0$, the quantity which fixes the state of the system is the thermodynamic potential Φ . The system, if not perturbed from outside reaches the state with minimum Φ .

At this point I note that Φ can have different expressions and names depending on what kind of parameters are fixed for the system. We are not interested here in different forms of Φ . We simply consider the Φ , which under the fixed parameters taken describes the system under study. If for somebody this general thermodynamic potential notion is too abstract, he may consider for example the fixed p, T, n = number of mols into the system case described by the Gibbs potential $G = F + pV$; $F = U - TS$; $dG = -SdT + Vdp - MdH + \sum_{i=1}^l \mu_i dn_i$; l = number of components in the system (here I considered also an external magnetic field H , M being the magnetisation).

I was mentioning that the minimum of Φ determines the state of the system. Here I must note that if this minimum is not the absolute minimum, but is a local minimum only, then the state of the system is metastable. If however the state is the equilibrium state, this is created by the absolute minimum (global minimum) of Φ . This is a stable state.

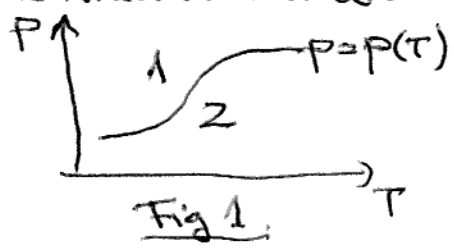
2. THE PHASE NOTION:

Let us consider for simplicity that one has a 1 component system ($l=1$). There are fixed independent parameter (e.g. P and T) values for which this system is perfectly homogeneous.

Now it happens that modifying the parameters (e.g. P, T) the system becomes divided in two homogeneous parts in contact in equilibrium with each other. However these two parts have the same chemical composition, their properties are different. These parts, or more precisely, these two states of the same matter which are able to exist in the same time in equilibrium, in contact with each other are named: "Phases". Both phases have their thermodynamic potential $\phi_1(T, p, n)$; $\phi_2(T, p, n)$, and being in contact in equilibrium it means

(1): $\phi_1(T, p, n) = \phi_2(T, p, n)$.

Since the number of moles n is fixed, mathematically Eq. (1) provides $p = p(T)$ which plotted in $(y = p, x = T)$ system of coordinates means a curve:



This curve is a phase separation curve. By crossing this curve we obtain a phase transition. In this lecture we will be interested in physical laws which drive the behavior of the system when we cross phase separation

curves. These laws are general physical laws. (e.g. $F = m \cdot a$ the Newton's law, when the non-relativistic classical mechanics is valid, is independent on what kind of body provides the mass m and who provides the force F). This means that the laws for which we are looking for here do not depend on what kind of system provides phase 1 and phase 2. These laws depend on the transition itself, and fix characteristics specific to the transition itself.

III. THE EHRENFEST'S CLASSIFICATION OF PHASE TRANSITIONS.

Paul Ehrenfest (1880-1933). Eq (1) shows that when we cross the phase separation line in Fig 1, the function ϕ remains continuous. Consequently, the thermodynamic function is always continuous in the parameter space of his variables, even if one encounters a phase transition. On that time only few things were known about phase transitions, hence Ehrenfest proposed a pure mathematical classification of phase transitions. He argued that ok, ϕ is a continuous function during phase transitions, but its derivatives are not obliged to satisfy this property. Hence we should concentrate on the ϕ derivatives and identify the lowest order derivatives which behave non-continuous (e.g. have a jump) at the phase transition. This lowest order provides a mathematical specification of the kind of the transition.

Consequently: e.g.: $dG = -SdT + Vdp - MdH + \sum_{i=1}^n \mu_i dn_i$ (2)

1^o First order phase transition: if the first order derivatives of ϕ possess a jump at the phase transition, e.g.

(3) $V = \left. \frac{\partial G}{\partial p} \right|_{T, H}$; $S = - \left. \frac{\partial G}{\partial T} \right|_{p, H}$ have jump at the transition ($\Delta V, \Delta S$)
 ΔV = first order transitions are accompanied by volume change
 $\Delta S = \frac{\Delta Q}{T}$: entropy variation is connected to ΔQ heat transfer (at constant t)

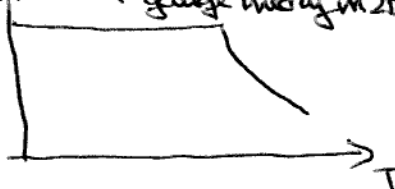
2° Second order phase transition: All first order derivatives of \mathcal{G} are continuous, but second order derivatives of \mathcal{G} possess jump at the transition. In this case $\Delta S = \Delta V = 0$, there is no latent heat, there is no volume change, but

$$\chi = \frac{\partial M}{\partial H} = - \frac{\partial^2 \mathcal{G}}{\partial H^2}; \quad c = T \frac{\partial S}{\partial T} = -T \frac{\partial^2 \mathcal{G}}{\partial T^2} \dots \quad (1)$$

i.e. the magnetic susceptibility, the specific heat have jump at the transition

3° Third order phase transitions: All first and second order derivatives of \mathcal{G} are continuous, only third order derivatives have a jump
(see D. F. Gross, E. Witten PRD21, 446 (1980); E.C. Erman et al. arXiv:1011.2732)

C ↑
Large N lattice gauge theory in 2D



The specific heat is continuous, but the first order derivative of C is discontinuous (= third order phase transition).

See also O.K. Rice, Phys. Rev. 93, 1161 (1954) for liquid He.

- T. Ma, S. Wang, AIP Advances 1, 042101 (2011)
- T. Eisele, Commun. Math. Phys. 90, 125 (1983) → in Random systems
- Third order phase transition, but symmetry breaking is not present
- For 4th order phase transit: S.D. Stojic et al. Physica A, 388, 1074 (2009) arXiv:cond-mat/0504161
(No cusp point, but jump in second derivative of χ)

∞° Infinite order phase transition: Kosterlitz-Thouless-Berezinskii transition. This is a topological phase transition: simple vortices → pair of vortices.

As can be seen, the Ehrenfest's classification is a simple attempt to say something about phase transitions on simple mathematical grounds. But this classification not provides in most cases main information about the physics which is going on. And we will be interested here in physics, and main physical properties at the transition. So we take notice here of the Ehrenfest's classification, but in our try to understand the physics, we do not follow it. On this line I note that today we know phase transitions which simply do not match the Ehrenfest's classification, as:

- a.) spin-glasses: cusp in χ , and broad maximum in C
(this is a some kind of "replica" ordering, which means ordering in time)
- b.) mixed order phase transitions (e.g. A. Bau et al. PRL 112, 015701 (2014))
order parameter discontinuous (as in I. order transitions), while correlation length diverges (as in II. order transitions). This here is given by an 1D model (Ising type) with long-range interaction. (arXiv:1309.4225)
- c.) correlation transitions (I. Danuka et al. Phys Rev E 58, 5403 (1998)): No jumps, but correlation functions behave completely different above and below T_0 . (1D Ising model with long-range interactions.)

d.) Change in topology transitions (called topological transitions) (A.A. Varlamov et al. *Advances in Phys* 38, 469 (1999)) (or see: topological insulators). There is not present latent heat (as in the case of I. order transitions), there is not present symmetry breaking (as in the case of II. order transitions). The topology of a closed surface (e.g. Fermi surface) changes.

III. Gibbs' Phase Rule :

I told that we are interested here in laws characterizing the transition itself (which do not depend on phase 1 and phase 2 placed in the neighborhood of the separation curve in Fig 1.). But you can ask me: do we have such laws et al? That such laws indeed exist (i.e. what the Theory of phase transitions aims, is indeed achievable) was shown for the first time by Gibbs in his phase Rule.

Let us have a system with l components (l different chemical components in $E_2(2)$). Furthermore let be m the number of macroscopic degrees of freedom of the system (= variance of the system = if we fix the components in the system (i.e. the n_i in the last term of dG in $E_2(2)$) how many independent variables remain in Φ : for example in $E_2(2)$ three: T, p, H , so $m=3$). The number m often is called to be the number of "macroscopic" interactions within the system (T : thermic; p : mechanic; H : magnetic; we could further add in dG the term $-DdE$, E = external electric field, D = dipole moment and in this case $m=4$; without external fields $m=2$).

In these conditions we are interested to find the maximum number of phases (r_{max}) which can exist in contact within the system. In order to find the answer, one realizes that the maximum number of phases in contact in the phase diagram (the space of independent parameters of Φ) must be placed around 1 point:

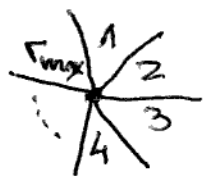


Fig 2.

Now each phase has its thermodynamic potential $\phi_1, \phi_2, \dots, \phi_{r_{max}}$; and since all these phases are in contact, all these ϕ_i must be equal. Equality between r_{max} functions means $r_{max}-1$ equations (equality between 2 functions means 1 equations). So we have the nr. of equations

$N_{eq} = r_{max}-1$; Now let us count how many parameters one has: from l components we obtain $l-1$ relative concentrations as parameters, and we further have m independent parameters in Φ . So the number of parameters that one has (= maximum number of unknown variables in the system);

$N_{par} = l-1+m$; Now in order to determine unanimously the parameters N_{par} we need $N_{par} = N_{eq}$ equations, i.e

(5): $r_{max}-1 = l-1+m \Rightarrow r_{max} = l+m \Rightarrow r \leq l+m$ (Gibbs' phase rule)

This relation as seen does not depend on what kind of phases are present, or what kind of material we analyse, so represents the kind of law which we are looking for. It is equally true for the phase transitions following the Big Bang, or the phase transitions in a steel or water.

As a particular case: when $l=1$ (one component system), and $m=2$ (system without external fields, so only 2 independent parameters are present, e.g. T, p), according to the Gibbs' phase rule: $r \leq 3$, so maximum a triple point can be present in the phase diagram.

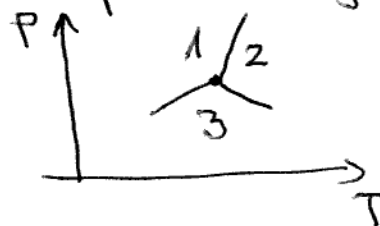


Fig 3.

I note that when the first supersolid theoretical paper has been sent to publication it contained a quadratic point in the phase diagram (at $l=1, m=2$). It was automatically rejected (after 2000). (Supersolid: coexistence of CDW with off-diagonal long range order = solid without friction.)

Possibility of the phase transitions in 1D.

Starting from this moment we leave the introductory part and analyse concretely the appearance possibilities of qualitative changes (i.e. phase transitions) in many-body systems, and we try to understand the physics behind them.

We start the study with the simplest possible case namely with 1D (one space dimension). I note that the deduction which you will follow originates from Landau.

Let us consider a 1D chain containing $N \gg 1$ particles, or sites in the localized case. We consider $N \gg 1$ because we have a many-body system whose treatment (in order to not be influenced by "surface" effects) must be done in the $N \rightarrow \infty$ thermodynamic limit (this means $N \rightarrow \infty, V \rightarrow \infty$, but $N/V = \text{constant}$.) We are interested to see if phase transitions can appear in such systems. In order to answer this question we suppose that from the existing N particles N_A there are present in the phase A, N_B in the phase B, and naturally: $N = N_A + N_B$; $N_A \gg 1$; $N_B \gg 1$; We are guided by the following observation: if two different phases A and B are possible to appear in the system, then we must have a situation when the two phases exist in contact in equilibrium within the system. Since the contact between 2 phases in 1D is a point, according to what we fixed in the introduction about the phase motion in the Introduction, this means the following configuration:

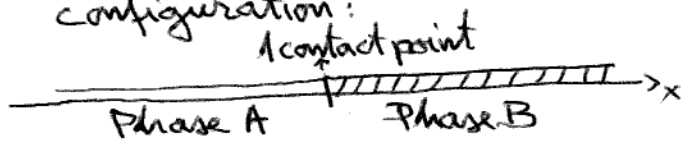


Fig 4.

In what will follow, we try to analyse if the configuration in Fig 4 can appear or not into the system. We consider N_A and N_B fixed but arbitrary.

If yes, than 2 different phases can be defined in the system under study, and this means that in this case one can has also a phase transition $A \rightarrow B$ (or $B \rightarrow A$) present in the system.

If not, than 2 different phases cannot be defined in the system, hence (if 2 phases cannot be defined) phase transition will not be present in the system.

If we watch the system in an arbitrary moment at an arbitrary value of the independent parameters of Φ , what we see is contained in Fig 5

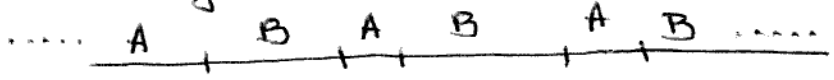


Fig 5.

If the thermodynamic potential of the phase A (at the fixed N_A) is Φ_A , and of the phase B (at the fixed N_B) is Φ_B , than

taking into account that Φ is an extensive (additive) quantity, and Φ_A not takes into consideration the phase B, nor Φ_B the phase A (i.e. the touching points not enter into the play), the thermodynamic potential of the system will be

(6): $\Phi_0 = \Phi_A + \Phi_B$; Note that this expression do not has information about the touching points.

Now let us take into consideration the touching points. Their number let be denoted by $n \ll N$, but because of the many-body nature $n \gg 1$. If we analyse Fig 5, we realize that there are no physical reasons to put a separation point exactly in the position specified in Fig 5: we can put them everywhere. This means that watched in time, the separation points in Fig 5 move. And a system of objects (here touching points) which move, in the language of a physicist means "gas". Consequently, from physical point of view, the studied situation represents in fact a "gas" (of contact points) embedded in the system described by Φ_0 in Eq. (6). Now we should analyze what kind of gas one has here. Since $n/N \ll 1$, the average distance between the component of the gas is high, so this gas is rare (diluted). Diluted gases are usually described as ideal gases: the average distance between the separation points being high, the interaction between them can be neglected. Otherwise, on physical grounds is difficult to define an interaction between contact points placed at high distance each from other, and also an interaction between the contact points and phases A or B. Consequently the gas that one have is a noninteracting gas (ideal gas) whose interaction with the medium in which it is immersed is also negligible. So the degrees of freedom of the ideal gas can be considered independent on the degrees of freedom of the medium possessing Φ_0 . This means that if Z_0 is the partition function of Φ_0 , and Z_1 is the partition function of the "contact points ideal gas" than the total partition function Z of the whole system is

(7): $Z = Z_0 \cdot Z_1$; (i.e. partition functions connected to independent degrees of freedom multiplies as independent probabilities.)

From statistical physics one knows that

$$(8) \quad \Phi = -k_B T \ln Z; \quad \Phi_1 = -k_B T \ln Z_1; \quad \Phi_0 = -k_B T \ln Z_0$$

where in order Φ, Φ_0, Φ_1 represent the thermodynamic potential of the whole system, of the starting system (without the contact points) and of the immersed ideal gas. Since fermionic or bosonic character cannot be attached to the contact points, the ideal gas that one has is a classical gas, hence

$$(9) \quad Z_1 = \frac{1}{n!} \left(\sum_i e^{-\frac{\epsilon_i}{k_B T}} \right)^n; \quad \text{where } \epsilon_i \text{ represents the possible energy values of a single touching point. Now}$$

$$(10) \quad \Phi = -k_B T \ln Z = -k_B T \ln(Z_0 \cdot Z_1) = \underbrace{-k_B T \ln Z_0}_{\Phi_0} - k_B T \ln Z_1 = \Phi_0 - k_B T \ln \left[\frac{1}{n!} \frac{N^n}{f(T,P)} \right]; \quad \text{where } \frac{N}{f(T,P)} = \sum_i e^{-\frac{\epsilon_i}{k_B T}} \text{ has been used as a notation (note: } N \text{ is fixed) but } N \text{ is arbitrary.}$$

Consequently, one obtains:

$$(11) \quad \Phi = \Phi_0 + k_B T \ln n! + n k_B T \ln \left[f(T,P) \frac{1}{N} \right];$$

At this moment one uses the Stirling formula $n! \cong \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$ ($\Gamma(z)$ continuously connects the factorials, the Stirling formula is coming from the high z expansion of $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt; \Gamma(n) = (n-1)!$; eg. $n=10; n! = 3.62 \cdot 10^6$; Stirling formula gives $3.60 \cdot 10^6$, so is rather good even starting from $n=10$). Using now (11), we find

$$(12): \quad \Phi = \Phi_0 + k_B T \frac{1}{2} \ln(2\pi n) + n k_B T \ln \frac{n}{e} + n k_B T \ln \left[f(T,P) \frac{1}{N} \right] = \Phi_0 + \frac{k_B T}{2} \ln(2\pi n) + n k_B T \ln \left[\frac{n}{N e} f(T,P) \right];$$

Since n/N appears in the argument of the $\ln(\dots)$ function, one further analyzes the thermodynamic potential per particle $\varphi = \frac{\Phi}{N}$ by dividing $E_2(12)$ by N . One finds

$$(13) \quad \varphi = \varphi_0 + \frac{1}{N} \frac{k_B T}{2} \ln(2\pi n) + \left(\frac{n}{N}\right) k_B T \ln \left(\frac{n}{N}\right) + \left(\frac{n}{N}\right) k_B T \ln \left[\frac{f(T,P)}{e} \right];$$

Now one takes the thermodynamic limit $N \rightarrow \infty$: the second term in $E_2(13)$ disappears, and $\frac{n}{N} = c$ can be introduced where c is the ideal gas concentration in the thermodynamic limit. One finds

$$(14) \quad \varphi = \varphi_0 + k_B T c \ln c + k_B T c \ln \left(\frac{f(T,P)}{e} \right); \quad c \ll 1;$$

Before going further, please note that the structure of the φ function which we obtained in $E_2(14)$ is intimately connected to the $T > 0$ situation. This structure completely disappears at $T=0$ and provides $\varphi = \varphi_0$. Consequently, the results will be connected to the $T > 0$ case. (It is also true, that at $T=0$, not φ but E decides the state of the system.)

Continuing the argumentation and denoting by $\psi_1(T, P)$ the quantity $\psi_1(T, P) = k_B T \ln \left(\frac{1}{\epsilon} f(T, P) \right)$, from (14) we find

$$(15) \quad \psi = \psi_0 + k_B T c \ln c + c \psi_1(T, P); \quad c \ll 1.$$

I introduce the following note here: From the theory of solutions we know that if we solve at a c value of concentration a material into a solvent, than the thermodynamic potential of the solution can be written in the form

$$(16) \quad \psi = \psi_0 + k_B T c \ln c + \sum_{j=1}^{\infty} c^j \psi_j(T, P);$$

Comparing Eqs. (15-16) we see that indeed $E_2(15)$ represents the thermodynamic potential of a dilute solution (the dilute solution of the contact points in the "solvent" described by ψ_0), i.e. the $c \rightarrow 0$ limit of $E_2(16)$.

Turning now back to $E_2(15)$, we have a $\psi(c)$ function, and we know that the state which minimizes this function (even in function of c) will become the state of the system. Hence one calculates

$$(17) \quad \frac{\partial \psi}{\partial c} = k_B T (1 + \ln c) + [\psi_1(T, P) + 2c\psi_2(T, P) + 3c^2\psi_3(T, P) + \dots]$$

where also further terms from $E_2(16)$ were considered. Now one takes the $c \ll 1$ (i.e. $c \rightarrow 0$ limit) in $E_2(17)$, and we find

$$(18) \quad \frac{\partial \psi}{\partial c} \approx k_B T \ln c; \quad c \ll 1 \Rightarrow \frac{\partial \psi}{\partial c} < 0;$$

$E_2(18)$ represents the main mathematical result of this deduction. Namely: The thermodynamic potential decreases if the concentration of the contact points increases. This means that the nature, in order to reach the equilibrium state in this studied case, will completely admix the two phases, and consequently, a situation presented in Fig 4. will never occur. As a consequence, a phase transition in the conditions of this deduction in 1D will never appear.

Analysis of the conditions of the deduction:

1° What is the ordered phase here?

If only 1 phase exists into the system, this must be the state which emerges at high T . This state is a completely disordered state. Since T is large now, we can state that the thermal fluctuations cause the disorder.

Our result deduced above state that other phases (in conditions of the deduction) cannot exist in 1D. But other phases, comparing them with the completely disordered phase, are ordered phases. So the result deduced above prohibits in 1D ordered phases. Precisely these prohibited ordered phases are the phases which look like the phase B in Fig 1. These are orderings which extend in the geometrical space over long distances. Such type of ordering is called "Long range order". Its exact definition can be formulated in terms of correlation functions, namely;

There exists a characteristic quantity $X(\vec{r})$ in the system, such that -10-

$$(19) \quad \lim_{|\vec{r}| \rightarrow \infty} \langle X(\vec{r}_0) X(\vec{r}_0 + \vec{r}) \rangle \neq 0 ;$$

If this property holds, long range order is present in the system.

2° Conditions under which the long range order is missing:

The study of the presented deduction shows that the following conditions have been used: i) $D=1$; ii) presence of the thermodynamic limit, iii) The contact points were considered non-interacting. This is rigorously true only if long-range interactions are not present. Consequently, short-range type of interactions were considered. iv) Since uniform T was considered, we are placed in a many-body thermodynamic system at non-zero temperature.

3° The rigorous statement:

In the light of the observations above the first rigorous statement obtained during our study can be formulated as follows:

In a $D=1$ dimensional many-body system at non-zero temperature in the presence of short-range interactions and in thermodynamic limit, long-range order cannot appear.

There exists a characteristic quantity $X(\vec{r})$ in the system, such that

$$(19) \quad \lim_{|\vec{r}| \rightarrow \infty} \langle X(\vec{r}_0) X(\vec{r}_0 + \vec{r}) \rangle \neq 0 ;$$

If this property holds, long range order is present in the system.

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3° The rigorous statement:

In the light of the observations above the first rigorous statement obtained during our study can be formulated as follows:

A) In a $D=1$ dimensional many-body system at non-zero temperature in the presence of short-range interactions and in thermodynamic limit, long-range order cannot appear.