Lecture 6:

I. Critical region, critical phenomena.

If somebody asks: where are valid the scaling properties, base on previous lectures we answer: close to continuous phase transiti lines. Indeed this is true. Critical regions are the parameter space regions where the scaling properties are valid, and critical phe-nomena are phenomena which take place in these regions. Hence, one can say that the $T \to T_c$ regions for continuous phase tran-sitions are critical regions. But the situation is that this is not the whole truth. There are also other critical regions which are not placed in the neighborhood of continuous phase transitions. The reason of this lecture chapter is to present to you also these regions.

We could ask now: what is needed in fact for a critical region? And at first view we are tempted to answer: an order parameter. This answer is not true. In fact, from mathematical point of view we need a diverging correlation length for the critical region. And it happens that one has regions where Landau order parameter cannot be defined because symmetry breaking is not present, but one has in that region a diverging correlation length, hence the region is a critical region.

A such domain is the domain placed in the neighborhood of a critical point of the first order phase transitions (see Fig 26.)

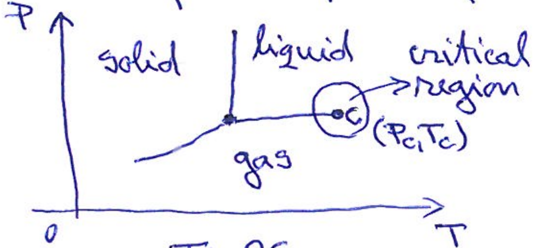

Fig 26.

I must underline that here a pure mathematical analogy is present without the emergence of a symmetry breaking. But this analogy allows to define all critical exponents and to satisfy all scaling laws (even does initially constructed for the order parameter Δ).

Indeed, in the vicinity of the critical point c (P_c, T_c) strong density-density correlations are present. If we reach point c the gas and liquid densities become equal, hence the density-density correlations extend to infinity and $\xi \to \infty$ occurs. Since density-density correlations are pre-sent, their correlation function $G(r,t)$ can be written, hence the ν, η critical exponents can be written as before (see (199, 200))

(273) $\xi \sim |t|^{-\nu}$; $G \sim \frac{1}{r^{d-2+\eta}}, t=0$; $t = \frac{T-T_c}{T_c}$ as before.

Specific heat always exists, so also α is defined as before (see (195))

(274) $C \sim |t|^{-\alpha}$.

Now the question is how we define β, γ, δ if order parameter in Landau sens cannot be defined? Mathematically, the role of the order parame-ter here is played by

(275) $\Delta = |\rho - \rho_c|$; since this continuously vary and becomes zero exactly at $t=0$.

Based on (275) the β exponent can be defined as

(276) $\Delta = |\rho - \rho_c| \sim |t|^{\beta}$; ρ_c being the density at the critical point c.

For the parameter ρ one needs external field type of parameter. Now the question is: who plays mathematically the role of the external field? In order to answer this question we compare a para-ferro transition (Fig 27a) with $|P-P_c|$ behavior around the point C (Fig 27b).

Fig 27

In Fig. 27a the spontaneous ferro phase is present only on the line below T_c, while in Fig 27b the $|P-P_c|$ behavior is seen near the critical point. The behavior is similar, hence $|P-P_c|$ plays matematically the role of h, consequently, the $h \sim \Delta^{\delta}$ relation is written here as: ($h = |P-P_c|$):

(277) $|P-P_c| \sim |\rho - \rho_c|^{\delta}$. This relation defines the critical exponent δ.

The last critical exponent is γ. For continuous phase transitions it is defined via: $\chi = \frac{\partial \Delta}{\partial h}$; $\chi \sim |t|^{-\gamma}$. But now, the compressibility plays the role of the susceptibility χ. Indeed

(278) $K = -\frac{1}{V}\frac{\partial V}{\partial P} = -\frac{1}{\frac{V}{m}} \cdot \frac{\partial \frac{V}{m}}{\partial P} = -\rho \frac{\partial \rho^{-1}}{\partial P}$; $\rho = \frac{m}{V}$ is the density

But one has

(279) $\frac{\partial \rho^{-1}}{\partial P} = -\frac{1}{\rho^2}\frac{\partial \rho}{\partial P} \Rightarrow K = \frac{1}{\rho}\frac{\partial \rho}{\partial P}$; hence

Furthermore $\partial(\rho - \rho_c) = \partial \rho$, $\partial(P-P_c) = \partial P$

(280) $K = \frac{1}{\rho}\frac{\partial[\rho - \rho_c]}{\partial[P-P_c]}$; In the same time $\rho - \rho_c = \mathrm{sign}(\rho-\rho_c)|\rho-\rho_c|$ and $P - P_c = \mathrm{sign}(P-P_c)|P-P_c|$; $\Delta = |\rho - \rho_c|$; $h = |P-P_c|$

(281) $K = \mathrm{sign}(\rho-\rho_c)\,\mathrm{sign}(P-P_c)\frac{\partial \Delta}{\partial h} \sim |t|^{-\gamma} \Rightarrow K \sim |t|^{-\gamma}$

So also γ was defined so we have all critical exponents at our disposal. And the critical exponents were proven to satisfy the scaling laws, so the scaling laws are present also in this region.

Two observations must be added here:

1.) From experimental point of view, at the beginning, was easier to measure at high precission the critical exponents around the point c than arount T_c for continuous phase transitions. Because of this reason the first high precision critical exponent values were comming from this field. Guggenheim measured several real gases (in 1945) as Ne, Ar, Kr, Xe, N_2, O_2, CO, CH_4, obtaining for all $\beta = \frac{1}{3}$. This was the first experimental proof of the universality, i.e. demonstrated that several properties which appear at qualitative changes are independent on system details.

2.) The critical region around C demonstrates that the scale invariance, and everything it introduces (static scaling, scaling laws) exceed considerably the frame of continuous phase transitions, and characterize in fact regions of diverging correlation length, i.e. qualitative changes in general. The Kadanoff block construction is a basic step in the deduction of the properties characterizing the critical region, and as such it will become the basic step also in the Renormalisation Group procedure which is today the heavy-artillery (at least one of heavy-artilleries) of the theory describing qualitative changes in many-body systems.

Goldstone mods.

Now we know the Ginsburg-Landau thermodynamic potential, scaling laws and critical exponents, so we can effectively calculate different things. First we will do this for concrete cases. About the Goldstone bosons I was mentioning some sentences before, around the Mermin-Wagner theorem, and now is the time to show mathematically how they appear. One must chose a case when a phase transition occurs such that a given continuous symmetry is broken. I have chosen the concrete case of an $O(n)$ symmetry. In this case the system is build up from localized spins which have n components as vectors. $O(n)$ is the continuous rotation symmetry for a system with such spins, the system being in the paramagnetic state. When this continuous symmetry breaks a ferromagnetic state appears as an ordered state.

In the studied concrete case $\Delta = M$ and we use the Ginsburg-Landau thermodynamical potential for the study (155)

$$(282)\quad \phi = \frac{1}{V}\int d^d r\left[\phi_0(\vec r) + AM^2(\vec r) + BM^4(\vec r) + g(\nabla M(\vec r))^2 - \vec h(\vec r)\vec M(\vec r)\right]$$

where $d=3$, $h=H$ external magnetic field.

I. The susceptibility expression:

First we analyze the magnetic susceptibility expression. In order to get the essence and not be lost in supplementary mathematical details we consider a homogeneous case first $M(\vec r) = M$, $h(\vec r) = h$, when (282) becomes

$$(283)\quad \phi = \phi_0 + AM^2 + BM^4 - \vec h \vec M$$

One considers that we are placed in the ordered phase $M \neq 0$, but $M \ll 1$, hence the leading term in (283) is the last term, so we have in fact

$$(284)\quad \phi = -\vec h\vec M = -\sum_{\alpha=1}^{n} h_\alpha M_\alpha$$

where the index α denotes the vector components.

Now let us calculate the susceptibility. One has

(285) $M_\alpha = -\frac{\partial \phi}{\partial h_\alpha}$; $\chi_{\alpha\beta} = \frac{\partial M_\alpha}{\partial h_\beta} = -\frac{\partial^2 \phi}{\partial h_\alpha \partial h_\beta}$

where susceptibility tensor $\chi_{\alpha\beta}$ shows how the α component of the magnetisation is influenced by the β component of the external field. Otherwise, the magnitude of the external field is

(286) $h = \left[\sum_{\alpha=1}^{n} h_\alpha^2\right]^{1/2}$; $\frac{\partial h}{\partial h_\alpha} = \frac{h_\alpha}{h}$;

and one has

(287) $\frac{\partial}{\partial h_\alpha} = \frac{\partial h}{\partial h_\alpha} \cdot \frac{\partial}{\partial h} = \frac{h_\alpha}{h} \frac{\partial}{\partial h}$

Now we have the possibility to calculate $\chi_{\alpha\beta}$. One has

$$\chi_{\alpha\beta} = -\frac{\partial}{\partial h_\alpha}\left[\frac{\partial}{\partial h_\beta}\phi\right] = -\frac{\partial}{\partial h_\alpha}\left[\frac{h_\beta}{h}\frac{\partial \phi}{\partial h}\right] = -\frac{\partial h_\beta}{\partial h_\alpha}\left(\frac{1}{h}\frac{\partial \phi}{\partial h}\right) -$$

(288) $$-h_\beta \frac{\partial}{\partial h_\alpha}\left[\frac{1}{h}\frac{\partial \phi}{\partial h}\right] = -\delta_{\alpha\beta}\frac{1}{h}\frac{\partial \phi}{\partial h} - \frac{h_\alpha h_\beta}{h}\frac{\partial}{\partial h}\left[\frac{1}{h}\frac{\partial \phi}{\partial h}\right] =$$

$$= -\delta_{\alpha\beta}\frac{1}{h}\frac{\partial \phi}{\partial h} - \frac{h_\alpha h_\beta}{h}\left[-\frac{1}{h^2}\frac{\partial \phi}{\partial h} + \frac{1}{h}\frac{\partial^2 \phi}{\partial h^2}\right] = -\frac{h_\alpha h_\beta}{h^2} \cdot \frac{\partial^2 \phi}{\partial h^2} -$$

$$\frac{1}{h}\frac{\partial \phi}{\partial h}\left(\delta_{\alpha\beta} - \frac{h_\alpha h_\beta}{h^2}\right) = e_\alpha e_\beta\left(-\frac{\partial^2 \phi}{\partial h^2}\right) + \left(\delta_{\alpha\beta} - e_\alpha e_\beta\right)\frac{M}{h} ;$$

In the above expression one considered in the last term $M = -\frac{\partial \phi}{\partial h}$ and one denoted $e_\alpha = \frac{h_\alpha}{h}$, hence $\sum_\alpha e_\alpha^2 = 1$. By this notation one has

(289) $\vec{h} = h\vec{e}$; $\vec{e} = (e_1, e_2, \dots e_n)$.

Based on (288) one sees that for $\alpha = \beta$ the last term cancels out and only the first term remains hence

(290) $\chi_{\parallel} = -\frac{\partial^2 \phi}{\partial h^2}$; $\chi_{\perp} = \frac{M}{h}$; notations can be introduced.

The first term in (290) shows how the magnetisation is modified by an external field which acts in its direction. This has always a finite value.

The second term however shows how an external field perpendicular to $\vec{M}$ acts on the magnetisation. Since we are in the ordered phase and M has nonzero value, it is important to underline that

(291) $\lim_{h \to 0} \chi_{\perp} = \infty$

This shows that our system is infinitely sensible (i.e. susceptible) on directions perpendicular to $\vec{M}$.

Let us analyse what is the reason for a such kind of behavior. First of all let us observe that when for spins a continuous rotation symmetry was present in the paramagnetic state, when $\vec{M}$ appears, it can chose between infinite many directions in order to fix himself as a vector.

That is, the ordered state, has a ground state which is infinitely degenerate in the vector space of the order parameter. Physically, all possible $\vec{M}$ directions are equivalent, so must exist a possibility to move the system with an infinitesimally small energy from one ground state to another ground state. This is produced by coherent transversal fluctuations which are the Goldstone mods, whose quasiparticles are the Goldstone bosons.

In other terms this looks as follows: When the $O(n)$ continuous symmetry is broken and the ordered phase with $\vec{M} \neq 0$ appears, this does not mean that fluctuations will not be present in the system. The fluctuations can be of two type:

i) longitudinal ones, which are parallel to $\vec{M}$. This modifies the length (magnitude) of M, consequently as seen from (283), these fluctuations modify ϕ hence they need a finite value of energy. (energy investment).

ii) transversal ones, these modify only the $\vec{M}$ direction, hence the first two terms in (283) are not modified, so ϕ remains almost unchanged. Hence extremely small energy investment is needed for this. These produce the Goldstone mods which appear inside the continuous symmetry broken ordered state (here at $d=3$). When the study is made via more detailed (282) where also the gradient term is present $(\nabla M)^2$, one observes that this term is smaller if the wavelenght is higher so $\lambda \to \infty$, i.e. $k = \frac{2\pi}{\lambda} \to 0$ will be prefered, with $\omega = \frac{2\pi c}{\lambda} \to 0$ (c is the velocity of the coherent fluctuations). Because of this reason the Goldstone mods will be characterized by the dispersion relation

(292) $k \to 0, \ \omega(k) \to 0$;

One more observation is made here, namely at $O(n)$ symmetry one has n vector components. So for 1 direction, one finds $n-1$ transversal directions, that is $n-1$ Goldstone mods are here present.

II. Detailed study.

Now we analyse the problem in more details based on (282) where also r-dependence is present. We will introduce r dependent fluctuations in it so the last term containing the external field is no more needed. So we use:

(293). $\phi = \frac{1}{V} \int d^d r \left[\phi_0(\vec{r}) + A M^2(\vec{r}) + B M^4(\vec{r}) + g(\nabla M(\vec{r}))^2 \right]$; $d=3$.

First let us fix the homogenious system which will be perturbed later by r-dependent fluctuations. One starts with $\vec{M}(\vec{r}) = \vec{m}$, hence the starting system is (from (293)):

(294) $\phi = \phi_0 + A m^2 + B m^4$; $m \neq 0$

ϕ has a minimum for the state of the system so $\frac{\partial \phi}{\partial m} = 0$ holds, i.e.

(295) $2Am + 4Bm^3 = 0 \Rightarrow m^2 = -\frac{A}{2B}$; ($A<0$; since $T<T_c$; $B>0$). see (143), (149).

The $\vec{m}$ vector is fixed, and let us take

(296) $\vec{m} = m\vec{e}$; $\vec{e} = (1,0,0,\dots 0)$; $\vec{e}$ = is the unit vector in the $\vec{m}$ direction.

At this moment one has

(297) $\vec{M} = \vec{m} = m\vec{e}$

Now we introduce $\vec{r}$ dependent fluctuations in the system parallel, and perpendiculare to $\vec{M}$ as follows

(298) $\frac{\vec{M}(\vec{r})}{m} = \vec{e} + \varphi_1(\vec{r})\vec{e} + \vec{\varphi}_\perp(\vec{r})$; $\vec{\varphi}_\perp = (0, \varphi_{\perp 1}, \varphi_{\perp 2}, \dots \varphi_{\perp n})$

where $\varphi_1(\vec{r})$ is a longitudinal fluctuation parallel to $\vec{M}$, and $\vec{\varphi}_\perp(\vec{r})$ is a transversal fluctuation perpendiculare to $\vec{M}$. Note that in fact:

(299) $\vec{M}(\vec{r}) = \vec{M} + m\varphi_1\vec{e} + m\vec{\varphi}_\perp$; $\vec{M} = m\vec{e}$

where one has

(300) $\delta M_{\parallel} = m\varphi_1$; $\delta M_\perp = m\varphi_\perp$; $\vec{M}(\vec{r}) = \vec{M} + \delta M_\parallel \vec{e} + \delta M_\perp \vec{e}_\perp$

where $\vec{e}_\perp$ is the unit vector in $\vec{\varphi}_\perp$ direction.

Now taking (299) for $\vec{M}(\vec{r})$ we have to calculate all components of (293), namely ϕ. Hence one has

(301) $M^2 = m^2\left[(1+\varphi_1(\vec{r}))^2 + \varphi_\perp^2(\vec{r})\right] = m^2(1 + 2\varphi_1(\vec{r}) + \varphi_1^2(\vec{r}) + \varphi_\perp^2(\vec{r}))$.

Furthermore, from (301):

(302) $M^4 = m^4(1 + 4\varphi_1(\vec{r}) + 2\varphi_1^2(\vec{r}) + 2\varphi_\perp^2(\vec{r}) + 4\varphi_1^2(\vec{r}) + O(\varphi^3))$

Now the $AM^2 + BM^4$ expression becomes:

$$AM^2 + BM^4 = Am^2(1 + 2\varphi_1 + \varphi_1^2 + \varphi_\perp^2) + Bm^4(1 + 4\varphi_1 + 2\varphi_1^2 + 2\varphi_\perp^2 + 4\varphi_1^2) =$$

(303) $$= (Am^2 + Bm^4) + 2\varphi_1 m^2\underbrace{(A + 2Bm^2)}_{=0} + \varphi_1^2 m^2(A + 6Bm^2) + m^2\varphi_\perp^2\underbrace{(A+2Bm^2)}_{=0}$$

$$= m^2(A + Bm^2) + m^2\varphi_1^2(A + 6Bm^2)$$

where we have taken into consideration that from (295) one has $A + 2Bm^2 = 0$. Furthermore, again from (295) $Bm^2 = -\frac{A}{2}$, so

(304)
$$(A + Bm^2) = A - \frac{A}{2} = \frac{A}{2},$$
$$(A + 6Bm^2) = A - 6\frac{A}{2} = -2A,$$

Then, from (304), by using $m^2 = -\frac{A}{2B}$ in the term not containing fluctuation

(305) $$\boxed{AM^2 + BM^4 = -\frac{A^2}{4B} - 2Am^2\varphi_1^2(\vec{r})\ ;}$$

As seen, and previously claimed, $\varphi_\perp$ not influences $AM^2 + BM^4$ term of ϕ.

Now the gradients follow. Since the longitudinal φ_1, as a vector, provides also longitudinal gradient, and the transversal $\varphi_\perp$ as vector provides also transversal gradient, from (299) one finds

(306) $(\nabla M)^2 = m^2\left[(\nabla\varphi_1)^2 + (\nabla\varphi_\perp)^2\right]$

Now adding (305, 306) in (293) and neglecting the constant in (305), taking into account that in fact $A<0$, one obtains:

$$\Delta\phi = m^2 \frac{1}{V}\int d^d r \left[2|A|\varphi_1^2(\vec r) + g\left((\nabla\varphi_1(\vec r))^2 + (\nabla\varphi_\perp(\vec r))^2 \right) \right] ; \quad (307)$$

choosing the energy units such to have $g=\frac{1}{2}$, one finds

(308): $$\Delta\phi = \frac{m^2}{2}\frac{1}{V}\int d^d r \left[(\nabla\varphi_1(\vec r))^2 + (\nabla\varphi_\perp(\vec r))^2 + 4|A|\varphi_1^2(\vec r) \right] ;$$

In order to simplify the expression, one takes now the Fourier transforms for φ_1 and $\varphi_\perp$:

(309) $$\varphi_1(\vec r) = \sum_{\vec k} e^{i\vec k\cdot\vec r}\varphi_{1\vec k} ; \quad \varphi_{\perp\alpha}(\vec r) = \sum_{\vec k} e^{i\vec k\cdot\vec r}\varphi_{\perp\alpha\vec k} ;$$ and has

(310) $$\nabla\varphi_1(\vec r) = \sum_{\vec k}(i\vec k)e^{i\vec k\cdot\vec r}\varphi_{1\vec k} ; \quad \nabla\varphi_{\perp\alpha}(\vec r) = \sum_{\vec k}(i\vec k)e^{i\vec k\cdot\vec r}\varphi_{\perp\alpha\vec k} ;$$

Now the squares can be calculated

$$\varphi_1^2(\vec r) = \sum_{\vec k,\vec k'} \varphi_{1\vec k}\varphi_{1\vec k'} e^{i\vec r(\vec k+\vec k')}$$

(311) $$(\nabla\varphi_1(\vec r))^2 = \sum_{\vec k,\vec k'} (i)^2 \vec k\cdot\vec k' \varphi_{1\vec k}\varphi_{1\vec k'} e^{i\vec r(\vec k+\vec k')}$$

$$(\nabla\varphi_\perp(\vec r))^2 = \sum_\alpha \sum_{\vec k,\vec k'} (i)^2 \vec k\cdot\vec k' \varphi_{\perp_\alpha\vec k}\varphi_{\perp_\alpha\vec k'} e^{i\vec r(\vec k+\vec k')}$$

Reintroducing these expressions in (308) we finds

(312) $$\Delta\phi = \frac{m^2}{2}\sum_{\vec k,\vec k'} \left[\varphi_{1\vec k}\varphi_{1\vec k'}(4|A|) + (i)^2\vec k\cdot\vec k'\varphi_{1\vec k}\varphi_{1\vec k'} + (i)^2\vec k\cdot\vec k'\sum_\alpha \varphi_{\perp_\alpha\vec k}\varphi_{\perp_\alpha\vec k'} \right] \times$$
$$\times\left(\frac{1}{V}\int d^d r\, e^{i(\vec k+\vec k')\vec r} \right) ;$$

But $\frac{1}{V}\int d^d r\, e^{i\vec r(\vec k+\vec k')} = \delta(\vec k+\vec k')$ hence $\vec k' = -\vec k$, then $(i)^2\vec k\vec k' = k^2$,
and $\sum_\alpha \varphi_{\perp\alpha\vec k}\varphi_{\perp\alpha\vec k} = \varphi_{\perp\vec k}^2$; Note that $\varphi_{\vec k} = \varphi_{-\vec k}$, because the homogenity and isotropy. Hence from (312) one finds

(313) $$\boxed{\Delta\phi = \frac{m^2}{2}\sum_{\vec k}\left[(k^2+4|A|)\varphi_{1\vec k}^2 + k^2\varphi_{\perp\vec k}^2 \right]} ;$$

The expression (313) hows how ϕ is modified given by the introduce fluctuations. Now we write (313) in a much more condensed form

First, based on (300) ($\delta M_\parallel = m\varphi_1$; $\delta M_\perp = m\varphi_\perp$) one denotes

(314) $$\delta M_{\parallel\vec k} = m\varphi_{1\vec k} ; \quad \delta M_{\perp\vec k} = m\varphi_{\perp\vec k} ;$$

Furthermore, we introduce the notation

(315) $$A^\parallel_{\vec k,\vec k'} = (k^2+4|A|)\delta_{\vec k,\vec k'} ; \quad A^\perp_{\vec k,\vec k'} = k^2\delta_{\vec k,\vec k'}$$

hence (313) becomes

(316) $$\Delta\phi = \frac{1}{2}\sum_{\vec k,\vec k'}\left(A^\parallel_{\vec k,\vec k'}\delta M_{\parallel\vec k}\delta M_{\parallel\vec k'} + A^\perp_{\vec k,\vec k'}\cdot\delta M_{\perp\vec k}\delta M_{\perp\vec k'} \right)$$

Now let us introduce $\sigma = \parallel,\perp$, and denote $i = (\sigma,\vec k)$; Furthermore let

$$\gamma_{ij} = A^\sigma_{\vec k,\vec k'}\delta_{\sigma,\sigma'} ; \quad y_i = \delta M_{\sigma\vec k} ; \quad (317)$$

In this condition (316) becomes (with Einstein convention)

(318) $\Delta\phi = \frac{1}{2}\gamma_{ij}\, y_i y_j$

Since such expressions have been intensively analized before (see (81) from lecture 3.) one knows that

(319) $\langle y_i y_j \rangle \sim \frac{\delta_{ij}}{\gamma_{ij}}$;

Using (317) one finds from here the longitudinal and transversal correlation functions in k-space:

(320) $G_{\parallel}(\vec{k},\vec{k}') = \langle \delta M_{\parallel\vec{k}}\, \delta M_{\perp\vec{k}'} \rangle \sim \frac{\delta_{\vec{k}\vec{k}'}}{k^2 + 4|A|}$; $G_{\parallel}(\vec{k},\vec{k}') = G_{\parallel}(\vec{k})\,\delta_{\vec{k},\vec{k}'}$

$G_{\perp}(\vec{k},\vec{k}') = \langle \delta M_{\perp\vec{k}}\, \delta M_{\perp\vec{k}'} \rangle \sim \frac{\delta_{\vec{k},\vec{k}'}}{k^2}$; $G_{\perp}(\vec{k},\vec{k}') = G_{\perp}(\vec{k})\,\delta_{\vec{k},\vec{k}'}$

Consequently:

(321) $\boxed{\begin{aligned} G_{\parallel}(\vec{k}) &\sim \frac{1}{k^2 + 4|A|} ; \\ G_{\perp}(\vec{k}) &\sim \frac{1}{k^2} ; \end{aligned}}$

Transforming back the correlation functions (321) in $\vec{r}$ space we can find their behaviour in the direct (geometrical) space.

Concerning $G_{\parallel}$, one knows that in d dimensions, the Fourier transform of $\frac{1}{k^2+x^2}$ (in our case $x = 2\sqrt{|A|}$) has the form

(322) $G(\vec{k}) \sim \frac{1}{k^2+x^2} \Rightarrow G(\vec{r}) \sim \frac{e^{-xr}}{r^{\frac{d-1}{2}}}$; (in our case $G_{\parallel}$ has such expression)

For our case d=3, and introducing the notation

(323) $\frac{1}{\xi} = x = 2\sqrt{|A|}$; $\xi = \frac{1}{2\sqrt{|A|}}$ we find

(324) $G_{\parallel}(\vec{r}) \sim e^{-\frac{r}{\xi}}$;

Consequently, the correlation function of the longitudinal fluctuation strongly decreases in space, hence these fluctuations have a short action radius (range), hence they cannot build up a collective mode, collective oscillations.

Contrary to this, the Fourier transform in d=3 of $G_{\perp}$ gives $\frac{1}{r}$ behavior. Indeed, since the $\sin \vec{k}\cdot\vec{r}$ term of the exponential $e^{i\vec{k}\cdot\vec{r}}$ gives an odd function integrated on a symmetric domain, hence cancels out, we remain with

(325) $G_{\perp}(\vec{r}) = \frac{1}{(2\pi)^3}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \int_0^{\Lambda} dk \cos[kr\cos\theta] = \frac{1}{4\pi r}$

where $\Lambda \to \infty$ was taken, z-axis was taken along the fixed $\vec{r}$ vector, and $\int_0^{\infty} \frac{\sin ax}{x} dx = \frac{\pi}{2}\,\mathrm{sign}(a)$ has been used. The $1/r$ decrease (as in the case of the Coulomb potential) signals a long-range behavior.

The long-range correlations between the transversal fluctuations build up a collective behavior which gives rise to the Goldstone mods.

Otherwise the pole of a correlation function in κ space gives the excitation spectrum. This is because if $G(\kappa)$ is infinitely large, will have contribution in all $G(r_{ij}=|\vec{r}_i-\vec{r}_j|)$. In our case, the $G_\perp$ has a pole at $\kappa\to 0$, signaling the $\omega(\kappa)|_{\kappa\to 0}\to 0$ spectrum of the Goldstone bosons (the quasiparticles of the collective transversal fluctuations. Since this $\omega(\kappa)$ dispersion relation is gapless at $\kappa\to 0$, and this is particle physics (or field theory) means masless, the Goldstone bosons are masless particles.

III Order parameter Stiffness. (rigidity)

If we analyze (290), $\chi_{\parallel}=-\frac{\partial^2\phi}{\partial h^2}$ we see that longitudinal sensibility (susceptibility) of the system is practically the same in the ordered and disordered state. Contrary to this the effect of transversal fluctuations is present only in the ordered state. Because of this reason the transversal term in (308) has a special importance in the ordered phase which is stressed by writing (308), namely

$$\Delta\phi=\frac{m^2}{2}\frac{1}{V}\int d^3r\left[(\nabla\varphi_1)^2+4|A|\varphi_1^2+(\nabla\varphi_\perp)^2\right]$$

in the form

$$(326)\quad \Delta\phi=\Delta\phi_{long}+\frac{1}{2}\int d^3r\, R\,(\nabla\varphi_\perp)^2;\quad R\sim m^2$$

where R = order parameter stiffness (usually $\sim m^2$, the square of the order parameter). $\Delta\phi_{long}$ being practically the same both in ordered and disordered phase is neglected from the discussion below.

Eq. (326) shows that an arbitrary modification attempt of the magnetization caused by a perturbation perpendicular to its orientation, will increase ϕ (i.e. will destabilize the system). Because of this reason, if a such type of perturbation occurs, the system reacts in such a way to restore his old state. This is made by <u>restoring forces</u> which we feel as stiffness. This stiffness appears in the ordered phase, and for long-range order (in the ordered phase), usually is proportional to Δ^2 (here m^2).

Because of the order parameter stiffness, stiffness measurements clearly indicate the position of a phase transition where a continuous symmetry is broken.

Otherwise, (326) has the analogy in the case of isotropic solid materials in the free energy expression (Einstein convention used)

$$(327)\quad \phi=\frac{1}{2}\int d^3r\left[\lambda\,\varepsilon_{ii}^2+2\mu\,\varepsilon_{ij}^2(\vec{r})\right];\quad 2\mu\to R;\ \mu=\text{shear modulus (modulus of rigidity)}$$

Here ε_{ij} = strain tensor, λ,μ = Lamé coefficients, and

$$(328)\quad \varepsilon_{ij}=\frac{1}{2}(\partial_i u_j+\partial_j u_i);$$ were $u_i(\vec{r})$ = displacement at the point $\vec{r}$ in i direction, ∂_j = derivative in j direction

The second term in (326) corresponds to the second term in (327), $\nabla\Psi_\perp$ corresponds to non-diagonal strain tensor component while the order parameter stiffness $\bar{K}$ corresponds to the shear modulus (Lamé coefficient μ). From (327) one knows that $\mu=0$ means fluid, while a rigid (i.e. solid) state is defined by $\mu\neq 0$. Otherwise the tangential component of the stress tensor σ is given by $\sigma_{ij}\sim\mu\,\varepsilon_{ij}$. The λ is the first Lamé parameter is related to the longitudinal (diagonal) components of the stress tensor σ via $\sigma_{ii}=\lambda\,\mathrm{tr}(\varepsilon_{ii})$, hence the λ term in (327) corresponds to $\Delta\emptyset_{long}$ in the equation (326).

IV. What we learn about critical exponents.

Since one has in (321) correlation functions, we can try to see what value of critical exponents (connected to the correlation function) they provide.

1.) The η exponent: It is defined as $G(r)\sim\frac{1}{r^{d-2+\eta}}$ at $t=0$ $(T=T_c)$ see (200). As seen, η is defined in geometrical $(\vec{r})$-space, while our deduced correlation functions are given in k-space. In order to make the connection we must transform $G(r)$ from (200) in k-space (by a Fourier transform).

$$(329)\quad G(\vec{k})=\int\frac{d^d r}{(2\pi)^d}\underbrace{\frac{1}{r^{d-2+\eta}}}_{G(\vec{r})}e^{-i\vec{k}\cdot\vec{r}}\ ,\quad r=|\vec{r}|;$$

e.g. for $d=3$ one has $f(\{\theta_\alpha\})=\sin\theta_1\,d\theta_1 d\theta_2$ and $\theta_2=\varphi$ holds.

Since $d^d r=f(\{\theta_\alpha\})\,r^{d-1}dr\prod_\alpha d\theta_\alpha$, and if one choses $\vec{r}$ (which is fixed in (329)) along de z (i.e. $\alpha=1$) axis: $\vec{k}\cdot\vec{r}=kr\cos\theta_1$, from (329) one findes $(k=|\vec{k}|,\ r=|\vec{r}|)$:

$$G(\vec{k})=\int dr\int\prod_\alpha d\theta_\alpha\frac{f(\{\theta_\alpha\})}{(2\pi)^d}\frac{r^{d-1}}{r^{d-2+\eta}}e^{-ikr\cos\theta_1}=$$

the variable change $x=kr$, $dr=\frac{1}{k}dx$ is made

$$(330)\quad =\int\frac{dx}{k}\int\prod_\alpha d\theta_\alpha\frac{f(\{\theta_\alpha\})}{(2\pi)^d}\left(\frac{x}{k}\right)^{1-\eta}e^{-ix\cos\theta_1}=$$

$$=\frac{1}{k^{2-\eta}}\underbrace{\left[\frac{1}{(2\pi)^d}\int dx\int\prod_\alpha d\theta_\alpha f(\{\theta_\alpha\})\,x^{1-\eta}e^{-ix\cos\theta_1}\right]}_{C\,=\,k\text{-independent constant}},\ \text{hence}$$

$$(331)\quad G(r)\sim\frac{1}{r^{d-2+\eta}}\Rightarrow G(k)\sim\frac{1}{k^{2-\eta}}$$

To deduce η, $T=T_c$, i.e $t=0$ must hold. But in (321), $A\sim t$, see (143), so at $t=0$ both correlation functions in (321) give

$$(332)\quad G_\parallel(k)\sim\frac{1}{k^2},\ G_\perp(k)\sim\frac{1}{k^2}\ ;\ t=0$$

Frome here, in the light of (331) it results

$$(333).\quad \boxed{\eta=0}$$

This result is related to ferromagnetism in $O(n)$ models treated with the Ginsburg-Landau theory. But since in (302) higher fluctuations orders have been neglected, it represents practically a mean field value. So, for further considerations let us fix that

(334). $\eta = 0$, for mean-field treatement

2) The ν critical exponent:

What we observe related to ν defined as $\xi \sim |t|^{-\nu}$ is that in (323) one has $\xi \sim \frac{1}{\sqrt{|A|}}$; $|A| \sim |t| \Rightarrow \xi \sim |t|^{-1/2}$, hence $\nu = \frac{1}{2}$. This ν value corresponds to a corelation function $G \sim \frac{1}{k^2 + \chi^2}$, where $\chi^2 \sim |A|$. In (321), G_{11} has such expression, but the treatement is given in the ordered phase. If we approach T_c from the disordered phase, there only G_{11} exists, so we obtain

(335) $\nu = \frac{1}{2}$

Since higher order fluctuations have been neglected in (302) (i.e. during the calculations) this is again a mean-field type of result. So one can state

(336) $\nu = \frac{1}{2}$, for mean-field treatement

V. Why appears a qualitative change in the system?

Since we are physicists, and it is known that the physics advances based on the question: why?, also during studying phase transitions, it comes the moment when we ask ourselves: why qualitative changes appear in physical systems? Now one has the sufficient background knowledge to answer this question at leas in the case of continuous phase transitions. I start by describing the

1. The Le Chatelier-Braun principle: Although formulated first in chemistry, has a much general applicability in nature. It states that: When any system at equilibrium becomes to be under the action of an external effect, the system reacts in a such a way to counteract the external effect and reestablish an equilibrium state. In short terms: whenever a system in equilibrium is disturbed, the system will respond by nullifying the external effect. This principle has been introduced in economy by Paul Samuelso in medicine and biology is called homeostasis, and in physics we observe this principle almost everyday, as for example by the restoring forces mentioned at the order parameter stiffness, curren that are started in thermodynamic systems in order to anihilate the action of different gradients, the Lenz's law in electrodynamics (the induced current direction is always such to oppose the change which produced it), etc.

This principle shows that each system react such to defend himself and to counteract external disturbancies. If you stay on the corridor, and a person is comming to you and starts to push you, you will react such to nullify this effect. How you do it, it depends on your personality (you go away or you hit over), but the final result will be the same, you will nullify the effect. Interestingly, this is not an exclusive human property, is present also in the lifeless dead matter, most precisely, the living matter hereditary obtained this property from the lifeless matter.

2. A reason for qualitative changes in many-body systems:

At the continuous phase transition the susceptibility diverges. This means that an infinitesimally small external action (perturbation or field) causes an infinitely large internal effect in the system. In such conditions the system is unable to defend himself Hence in order to have the possibility to defend himself, changes its quality.

We also know that this reason for qualitative changes exceeds considerably the frame of continuous phase transitions. For example the poles of the vertex function in many-body physics signal qualitative changes in the system. The vertex function in fact represents the two-body interaction inside the many-body system, and if it diverges, that means that the system is unable to screen And without screening the system cannot defend himself, hence in order to have defense possibilities, changes his quality.