Lecture 7:

The aim of this lecture is to deduce the mean-field critical exponent in a concrete case. The simplest example for this is a localised spin system which is treated in mean-field approximation.

I. Free spin system under magnetic field.

The Hamiltonian that one has in this case is simply

(337) $\hat{H}_0 = -g\mu_B \vec{H}\cdot\sum_{i=1}^{N}\vec{S}_i$; and what we do is to calculate the induced so non-spontaneous magnetisation of the system.

Since $\vec{H}$ is homogenous and constant, one turns the z axis in $\vec{H}$ direction, obtaining $H = |\vec{H}| = H_z$, and the Hamiltonian (337) becomes

(338) $\hat{H}_0 = -g\mu_B H\sum_{i=1}^{N} S_i^z$; Taking the most general case with quantum spins, S_i^z can have $2S+1$ values in between $-S$, and $+S$, S being the spin value.

The partition function becomes:

$$Z = \mathrm{Tr}\, e^{-\beta\hat{H}_0} = \sum_{m_1=-S}^{+S}\sum_{m_2=-S}^{+S}\dots\sum_{m_N=-S}^{+S} e^{[\overbrace{\beta g\mu_B H}^{x}(m_1+m_2+\dots+m_N)]} =$$

(339)

$$= \Big[\sum_{n=-S}^{+S} e^{xn}\Big]^N = \Big[e^{-xS} + e^{-x(S-1)} + e^{-x(S-2)} + \dots + e^{-x(S-2S)}\Big]^N ; \quad x = g\mu_B\beta H$$

$$= \Big\{e^{-Sx}\big[1+e^{x}+e^{2x}+\dots+e^{2Sx}\big]\Big\}^N = \Big\{e^{-Sx}\sum_{n=0}^{2S} y^n\Big\}^N = \Big\{e^{-Sx}\frac{y^{2S+1}-1}{y-1}\Big\}^N ; \quad y = e^x$$

$$= \Big\{\frac{e^{Sx+x}-e^{-Sx}}{e^x-1}\Big\}^N = \Big\{\frac{e^{\frac{x}{2}}\big[e^{Sx+\frac{x}{2}} - e^{-Sx-\frac{x}{2}}\big]}{e^{\frac{x}{2}}\big[e^{\frac{x}{2}}-e^{-\frac{x}{2}}\big]}\Big\}^N = \Big\{\frac{\sinh[(S+\frac12)x]}{\sinh\frac{x}{2}}\Big\}^N ;$$

where $\sinh z = \frac{e^z - e^{-z}}{2}$; Consequently, the partition function in the most general quantum case becomes

(340) $Z_Q = \Big[\frac{\sinh[(S+\frac12)x]}{\sinh\frac{x}{2}}\Big]^N$; $x = g\mu_B\beta H$; N = nr. of sites in the system. $\beta = \frac{1}{k_B T}$;

The Ising case: In the partition function Z, $m_i = \pm S$, since in this case $S^z = \pm S$, hence

(341): $Z = \Big[\sum_{n=\pm S} e^{xn}\Big]^N = \Big[e^{xS} + e^{-xS}\Big]^N = \Big[2\cdot\frac{e^{xS}+e^{-xS}}{2}\Big]^N = 2^N\cosh^N(Sx)$

where $\cosh z = \frac{e^z+e^{-z}}{2}$; Then, in the Ising case the partition function becomes

(342): $Z_I = \big[2\cosh(Sx)\big]^N$; $x = g\mu_B\beta H$

The classical case: In the classical case $\vec{H}\cdot\vec{S} = HS\cos\theta$ where θ can have continuously arbitrary value. Consequently, in this case

(343): $\Big[\sum_n e^{xn}\Big] \to \underbrace{\frac{1}{4\pi}\int d\Omega}_{\sum_n} \cdot e^{\underbrace{g\mu_B\beta HS}_{xS}\cos\theta}$; Ω = solid angle, $d\Omega = \sin\theta\, d\theta\, d\varphi$

Consequently: (see fig 28 for polar coordinates):

$$(344)\quad Z_c = \left[\frac{1}{4\pi}\int e^{xS\cos\theta}\, d\Omega\right]^N = \left[\frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, e^{xS\cos\theta}\sin\theta\right]^N ; \text{ But}$$

$$(345)\quad \int_0^{\pi} e^{xS\cos\theta}\sin\theta\, d\theta = \int_1^{-1} e^{xSt}(-dt) = \int_{-1}^{1} e^{xSt}\, dt = \frac{1}{xS}\, e^{xSt}\Big|_{-1}^{1} = \frac{1}{xS}\left[e^{xS} - e^{-xS}\right]$$

where $t = \cos\theta$; $dt = -\sin\theta\, d\theta$; and

$$e^{xS} - e^{-xS} = 2\sinh(xS)$$

Furthermore since is not present φ in the function which is integrated: $\int_0^{2\pi} d\varphi = 2\pi$, and hence turning back to (344) we find

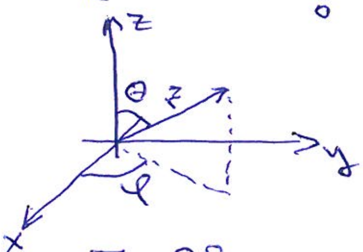

Fig 28

$$Z_c = \left[\frac{1}{4\pi}\cdot 2\pi \cdot \frac{2}{xS}\sinh(xS)\right]^N = \left[\frac{\sinh(xS)}{xS}\right]^N$$

consequently:

$$(346)\quad Z_c = \left[\frac{\sinh(xS)}{xS}\right]^N ;$$

One sees that xS enters in Z_I, Z_c, hence also xS dependence can be introduced in Z_Q obtaining

$$(347)\quad Z_Q = \left[\frac{\sinh \frac{2S+1}{2S}\cdot xS}{\sinh \frac{xS}{2S}}\right]^N ;$$

Introducing now the notation $\bar{x} = xS = g\mu_B\beta HS$, we find the partition functions of non-interacting spins in external field as:

$$(348)\quad Z_Q = \left[\frac{\sinh\frac{2S+1}{2S}\bar{x}}{\sinh\frac{\bar{x}}{2S}}\right]^N ; \quad Z_I = \left[2\cosh\bar{x}\right]^N ; \quad Z_c = \left[\frac{\sinh\bar{x}}{\bar{x}}\right]^N$$

Now $\phi = -k_BT\ln Z$, $M = -\frac{\partial\phi}{\partial H}$ is the root for obtaining the magnetisation M. The thermodynamical potentials become:

$$(349)\quad \begin{aligned} \phi_Q &= -k_BTN\ln\left[\frac{\sinh\frac{2S+1}{2S}\bar{x}}{\sinh\frac{\bar{x}}{2S}}\right]; \\ \phi_I &= -k_BTN\ln\left[2\cosh\bar{x}\right]; \qquad \bar{x} = g\mu_B\beta HS \\ \phi_c &= -k_BTN\ln\left[\frac{\sinh\bar{x}}{\bar{x}}\right]; \end{aligned}$$

Now from (349) the magnetisation is obtained as a simple derivative of (349) in function of H. Since the sinh and cosh derivatives are simple: $(\sinh x)' = \cosh x$; $(\cosh x)' = \sinh x$, one finds:

$$(350)\quad \begin{aligned} M_Q &= g\mu_BSN\left[\frac{2S+1}{2S}\coth\left(\frac{2S+1}{2S}\bar{x}\right) - \frac{1}{2S}\coth\left(\frac{\bar{x}}{2S}\right)\right] = g\mu_BSN\, B_S(\bar{x}) \\ M_I &= g\mu_BSN\cdot\tanh\bar{x}; \\ M_c &= g\mu_BSN\left[\coth\bar{x} - \frac{1}{\bar{x}}\right] = g\mu_BSN\, L(\bar{x}); \end{aligned}$$

where $\bar{x} = g\mu_B\beta HS$, $\tanh z = \frac{\sinh(z)}{\cosh(z)}$; $\coth z = \frac{1}{\tanh z} = \frac{\cosh z}{\sinh z}$;

Furthermore, two functions have been introduced, namely:

The Brillouin-function

$$B_S(\bar{x}) = \frac{2S+1}{2S} \coth\left(\frac{2S+1}{2S}\bar{x}\right) - \frac{1}{2S}\coth\left(\frac{\bar{x}}{2S}\right); \quad (351)$$

and the Langevine function, namely:

$$L(\bar{x}) = \coth \bar{x} - \frac{1}{\bar{x}}; \quad (352)$$

Now we simplify the notation to $\mu = g\mu_B S$; $\bar{x} = \mu\beta H$, and obtains the magnetisations as

(353). $M_Q = \mu N B_S(\mu\beta H)$; $M_I = \mu N \tanh(\mu\beta H)$; $M_c = \mu N L(\mu\beta H)$

Now we observe that taking $S \to \infty$, $\sinh \frac{\bar{x}}{2S} \to \frac{\bar{x}}{2S}$; $\cosh \frac{\bar{x}}{2S} \to 1$ one finds

(354) $\lim_{S\to\infty} B_S(\bar{x}) = L(\bar{x})$

Furthermore for quantum spin in $S=\frac{1}{2}$ case we have only $S^z = \pm S$ z-spin projections, hence for $S=\frac{1}{2}$, one has

(355) $B_{\frac{1}{2}}(\bar{x}) = 2\coth(2\bar{x}) - \coth\bar{x} = 2\frac{\cosh 2\bar{x}}{\sinh 2x} - \frac{\cosh\bar{x}}{\sinh\bar{x}}$

Using the fact that $\sinh 2\bar{x} = 2\sinh\bar{x}\cosh\bar{x}$, $\cosh 2\bar{x} = \cosh^2\bar{x} + \sinh^2\bar{x}$, we find

(355) $B_{\frac{1}{2}}(\bar{x}) = \frac{\cosh^2\bar{x} + \sinh^2\bar{x}}{\cosh\bar{x}\sinh\bar{x}} - \frac{\cosh\bar{x}}{\sinh\bar{x}} = \frac{\sinh^2\bar{x}}{\cosh\bar{x}\sinh\bar{x}} = \frac{\sinh\bar{x}}{\cosh\bar{x}} = \tanh\bar{x}$

Consequently

(356). $\lim_{S\to\frac{1}{2}} B_S(\bar{x}) = \tanh\bar{x}$

As seen, the quantum case reduces to classical case in $S\to\infty$ limit (from Quantum Mechanics one knows that the infinite large quantum numbers mean classical limit), and the quantum case reduces to the Ising case for $S=\frac{1}{2}$. Hence in fact the Brillouin function contains in itself both the Ising and classical limits, consequently below we analyse only the quantum case

(357) $M_Q = \mu N B_S(\mu\beta H)$; $\mu = g\mu_B S$; $\beta = \frac{1}{k_B T}$.

II. Interaction treated in mean-field approximation:

If interaction is present between the spins, the Hamiltonian $\hat{H}$ will contain besides $\hat{H}_0$ from (337) also an interaction term $\hat{H}_I$, so

(358) $\hat{H} = \hat{H}_0 + \hat{H}_I$, which we take in Heisenberg form

(359) $\hat{H}_I = -\sum_{\langle ij\rangle} J_{ij}\vec{S}_i\cdot\vec{S}_j$; $J_{ij} = J > 0$ (ferromagnetic coupling), and $\langle ij\rangle$ = nearest neighbors.

In mean-field the true inter-particle interaction is substituted by the one-particle interaction with a mean field, hence:

(360) $\vec{S}_i\cdot\vec{S}_j \to \vec{S}_i\langle\vec{S}_j\rangle + \vec{S}_j\langle S_i\rangle \to 2\vec{S}_i\cdot\langle\vec{S}_j\rangle$

where the last step is because the homogenity of the system. Hence:

(361) $\hat{H}_I = -2J\sum_{\langle ij\rangle}\vec{S}_i\langle\vec{S}_j\rangle$; But now one uses:

(362) $\sum_{\langle ij\rangle} = \frac{1}{2}\sum_i\sum_{j\in nn(i)}$; where $\frac{1}{2}$ as prefactor avoids double-counting. Considering $\langle\vec{S}_j\rangle$ the same on all sites and denoting by z = nr. of nearest-neighbors, one finds:

(363) $\hat{H}_I = -Jz \sum_i \vec{S}_i \langle \vec{S}_j \rangle$; where $\langle \vec{S}_j \rangle$ = is the same on all sites

Now taking into consideration that the magnetisation is

(364) $\vec{M}_Q = g\mu_B \sum_{i=1}^{N} \langle \vec{S}_i \rangle = g\mu_B N \langle \vec{S}_j \rangle$; $\langle \vec{S}_j \rangle = \frac{\vec{M}_Q}{g\mu_B N} = \frac{\vec{m}}{g\mu_B}$;

where

(365) $\vec{m} = \frac{\vec{M}_Q}{N} = \vec{m}_Q$. Turning now back to (363), in mean-field approximation one finds

(366): $\hat{H}_I = -\frac{Jz}{g\mu_B} \vec{m}_Q \sum_i \vec{S}_i$; Introducing the notation $\lambda = \frac{Jz}{(g\mu_B)^2}$, one finds

(367): $\hat{H}_I = -g\mu_B(\lambda \vec{m}_Q) \sum_i \vec{S}_i$; with this result, the full Hamiltonian becomes

(368)
$$\hat{H} = \hat{H}_I + \hat{H}_0 = -g\mu_B(\vec{H} + \lambda\vec{m})\sum_i \vec{S}_i = -g\mu_B \vec{H}_{eff} \sum_i \vec{S}_i$$
$$\vec{H}_{eff} = \vec{H} + \lambda\vec{m} \ ; \ \vec{m} = \vec{m}_Q = m_{quantum}, \ \lambda = \frac{Jz}{(g\mu_B)^2}$$

As seen from (368), the only change is that instead of $\vec{H}$, now the field $\vec{H}_{eff} = \vec{H} + \lambda\vec{m}$ acts on the spins. Since $\langle \vec{S}_i \rangle$ is considered parallel to $\vec{H}$, hence $\vec{m}$ is in the direction of $\vec{H}$ and (388) can be treated as a scalar relation

(369) $H_{eff} = H + \lambda m$; is the effective magnetic field.

<u>III The equation for magnetisation.</u>

Because of (369), the unique modification in (353) is that instead of H, one has $H + \lambda m$ in the Brillouin function. Since $M_Q/N = m$, we obtain the following equation for the magnetisation per particle m:

(370) $m = \mu B_S[\mu\beta(H + \lambda m)]$; $\mu = g\mu_B S$; $\lambda = \frac{Jz}{(g\mu_B)^2}$; $\beta = \frac{1}{k_B T}$

g = Landé g factor, μ_B = Bohr magneton

Often λ is called the Weiss molecular field coefficient.

<u>1) The critical temperature:</u>

We are looking for $m \neq 0$ solutions of (370) at $H = 0$. The studied equation becomes:

(371) $m = \mu B_S[\mu\lambda\beta m]$

Since at $T \cong T_c$ $m \ll 1$, we need the low argument expansion of $B_S(x)$ which, based on the $x \to 0$ expression of $\coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \dots$ becomes

(372) $B_S(x)\Big|_{x\to 0} = \frac{S+1}{3S}x - \frac{S+1}{3S}\cdot\frac{2S^2+2S+1}{30S^2}\cdot x^3 + \dots$

To be complete, I also provide the $x \to 0$ expansion for classical spin (Langevine function) and Ising spin (hyperbolic tangent):

(373)
$$L(x)\Big|_{x\to 0} = \frac{x}{3} - \frac{x^3}{45} + \dots$$
$$\tanh x\Big|_{x\to 0} = x - \frac{x^3}{3} + \dots$$

Note that in the opposite limit $x \to \infty$; ($\lim_{x\to\infty} \coth x = \lim_{x\to\infty} \tanh x = 1$) one has

(374) $\lim_{x\to\infty} B_S(x) = \lim_{x\to\infty} L(x) = \lim_{x\to\infty} \tanh(x) = 1$; the functions being odd in x hence $x \to -\infty$ provides -1.

For $x \to 0$, the Brillouin function in (371), based on (372) becomes

(375) $B_S(\mu\lambda\beta m)\Big|_{m\to 0} = \frac{S+1}{3S}\mu\lambda\frac{1}{K_BT}\cdot m + O(m^3)$;

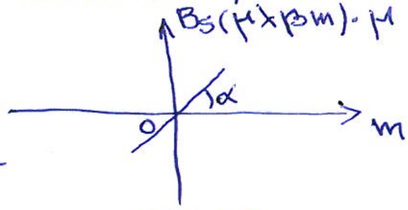

Fig 29

So the derivative of B_S in function of m around the origine becomes a line with slope

(376): $tg\,\alpha = \frac{S+1}{3S}\mu\lambda\frac{1}{K_BT}\cdot\mu$;

see Fig 29. Note that this slope decreases, if T increases. Now, based on (374, 376) the full plot of the $\mu B_S(\mu\lambda\beta m)$; (the right side of (371)) becom

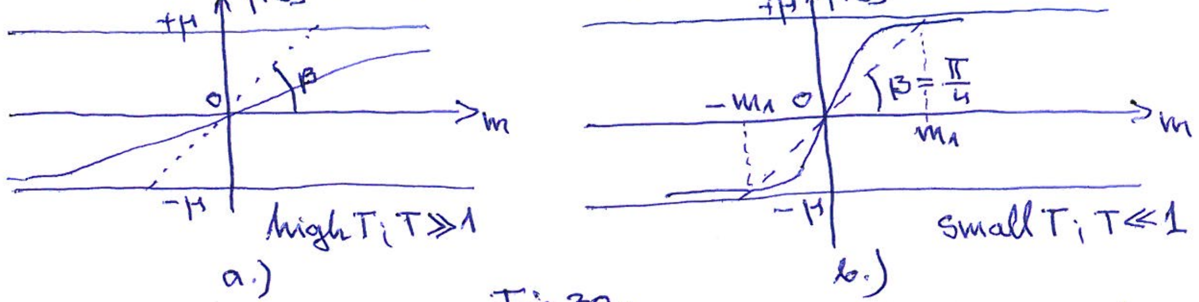

Fig 30.

as shown in Fig 30. The dotted line represents the $f(m) = m$ function present in the left side of (371). This is a line, constant in temperature, with $\beta = \frac{\pi}{4}$, hence constant slope $tg\,\beta = 1$.

Now if $T \gg 1$ (Fig 30a) the $f(m)$ and $g(m) = \mu B_S(\mu\lambda\beta m)$ functions have only one intersection, namely at $m = 0$, consequently, at $T \gg 1$ high, spontaneous magnetisation does not exists, we are placed in the paramagnetic state.

Contrary to this, if $T \ll 1$ (i.e. is small, see Fig 30b), between the $f(m)$ and $g(m)$ functions now three intersection points occur, namely besides the trivial (and paramagnetic) $m = 0$, one has also intersection (graphical solution of (371) at $m = \pm m_1 \neq 0$. This signals that we are placed inside the ferromagnetic phase. When the ferromagnetism occurs the system choses $+m$ or $-m$ value depending on the momentum local action on it. The transition occurs at $\alpha = \beta$ hence

(377) $\frac{S+1}{3S}\mu^2\lambda\frac{1}{K_BT_c} = 1 \Rightarrow T_cK_B = \frac{S+1}{3S}\mu^2\lambda = \frac{S+1}{3S}\underbrace{(g\mu_BS)^2}_{\mu}\overbrace{\left(\frac{Jz}{(g\mu_B)^2}\right)}^{\lambda} = \frac{S(S+1)}{3}Jz$;

Consequently

(378) $K_BT_c = Jz\frac{S(S+1)}{3}$; or $K_BT_c = \frac{S+1}{3S}\lambda\mu^2$;

2.) The order parameter close to T_c:

The order parameter equation is (371), namely

$m = \mu B_S(\mu\lambda\beta m)$; (i.e. (371)) ; $\Delta = m$;

In order to find the order parameter equation in explicite T dependent form we must use the B_S Taylor expansion in (372) up to third order:

(379) $m = \frac{S+1}{3S}\mu^2\lambda\beta m - \frac{S+1}{3S}\cdot\frac{2S^2+2S+1}{30S^2}\mu^4(\lambda\beta m)^3 + \dots$

Based on (378), the first term in the right side is $\frac{T_c}{T}m$; hence simplifying by m in (379) we obtain

(380) $1 - \frac{T_c}{T} = -\left[\frac{S+1}{3S}\cdot\frac{2S^2+2S+1}{30S^2}\mu^4\lambda^3\beta^3\right]m^2$

Since $1 - \frac{T_c}{T} = \frac{T-T_c}{T} = t$; and $t < 0$ below T_c, from (380) in $T \to T_c$ limit (i.e. in the right side $\beta \to \beta_c$ can be taken which is smooth and nonzero around T_c

one obtains

(381) $|t| = C\cdot m^2$; $C = \frac{S+1}{3S}\cdot\frac{2S^2+2S+1}{30S^2}\frac{\mu^4\lambda^3}{(k_BT_c)^3}$; Based on (378) one has $(k_BT)^3 = \left(\frac{S+1}{3S}\right)^3\lambda^3\mu^6$

we obtain for c the expression

(382) $C = \frac{9}{10}\frac{2S^2+2S+1}{(S+1)^2}\frac{1}{\mu^2}$; $\sqrt{C} = \frac{3}{\mu\sqrt{10}}\frac{\sqrt{2S^2+2S+1}}{S+1} = \frac{3}{\mu\sqrt{10}}\left[1+\frac{S^2}{(S+1)^2}\right]^{1/2}$;

and we get

(383) $m = \frac{1}{\sqrt{C}}|t|^{\frac{1}{2}}$; $T\to T_c$; $T<T_c$; or $\frac{m}{\mu} = |t|^{1/2}\frac{\sqrt{10}}{3}\frac{1}{\sqrt{1+\frac{S^2}{(S+1)^2}}}$; $\mu = g\mu_B S$

From where is seen that

(384): $\boxed{\beta = \frac{1}{2}}$; Note that the function $m(T)$ has infinite slop at $T = T_c$.

3) The stability analysis: We obtained in (383) the order parameter close to T_c. This expression has been obtained from the order parameter equation (371). But the order parameter equation, as shown in Fig 30, even below T_c allows also the trivial solution $m=0$ (and note, this is always so for any kind of order parameter equation everywhere).
Consequently;

(384) $T<T_c$; $m \begin{cases} m=0 \\ m\neq 0\end{cases}$ } two solutions exist

That physically what solution appears is determined by ϕ, the thermodynamic potential. That is why in the $m\neq 0$ region always also a stability analysis must be performed in order to see if at a given $T<T_c$ the $m=0$ or $m\neq 0$ solution has the minimum ϕ. For this reason we turn back to (349) from where we take $\phi = \phi_{cl}$ (quantum case), denote by $\varphi = \frac{\phi}{N}$ the thermodynamic potential per particle, use the $\bar{x} = \mu\beta(\lambda m)$ notation (note that $H\to H_{eff} = H+\lambda m$, and the external field $H=0$) obtaining:

(385) $\varphi = -k_BT\ln\frac{\sinh\left[\frac{2S+1}{2S}(\mu\beta\lambda m)\right]}{\sinh\left[\frac{\mu\beta\lambda m}{2S}\right]}$;

For the paramagnetic ($m=0$) phase, from (385) one obtains (using:

(386) $\sinh x\big|_{x\to 0} = x + \frac{x^3}{3!}+\dots$

with only the first term), one finds:

(387) $\varphi_{(m=0)} = \varphi_0 = -k_BT\ln(2S+1)$;

Now close to T_c, using (386) and simplifying by $(\mu\beta\lambda m)$ one finds

(386) $I = \frac{\sinh\left[\frac{2S+1}{2S}(\mu\beta\lambda m)\right]}{\sinh\left[\frac{\mu\beta\lambda m}{2S}\right]} = \frac{\frac{2S+1}{2S}+\frac{1}{6}\left(\frac{2S+1}{2S}\right)^3(\mu\beta\lambda m)^2}{\frac{1}{2S}\left[1+\frac{S}{3}\left(\frac{1}{2S}\right)^3(\mu\beta\lambda m)^2\right]} = \left[(2S+1)+\frac{1}{6}\frac{(2S+1)^3}{(2S)^2}(\mu\beta\lambda m)^2\right]\times$

$\times\left[1-\frac{S}{3}\left(\frac{1}{2S}\right)^3(\mu\beta\lambda m)^2\right]$; where for $z\ll 1$, $\frac{1}{1+z}\cong 1-z$ has been used.

Multiplying the two terms in the right side, one obtains (by neglecting four order terms:

(387). $I = (2S+1)\left[1+\frac{S(S+1)}{6S^2}(\mu\beta\lambda m)^2 + O(m^4)\right]$;

Hence φ from (385) becomes for $m\ll 1$ of the form:

(388) $\varphi = -k_BT\ln(2S+1) - k_BT\ln\left[1+\frac{S(S+1)}{6S^2}(\mu\beta\lambda m)^2 + O(m^4)\right] =$
$= \varphi_0 - k_BT\ln\left[1+\frac{S(S+1)}{6S^2}(\mu\beta\lambda m)^2 + O(m^4)\right]$; This means that $\Delta\varphi = \varphi_{(m)} - \varphi_0$

(389) $\Delta\varphi = -k_BT\ln\left[1+\frac{S(S+1)}{6S^2}(\mu\beta\lambda m)^2 + O(m^4)\right]$; and $\Delta\varphi < 0$ is needed for the stability of the ordered phase

Because at $x>0 \Rightarrow \ln(1+x) > 0 \Rightarrow$

(390) $\Delta\varphi = \varphi_{(m)} - \varphi_0 < 0 \Rightarrow \varphi_{(m)} < \varphi_0$; Consequently, when the $m\neq 0$ phase appears, it is energetically stable

hence indeed the solution (383) exists, and $\beta = \frac{1}{2}$

What about the stability for arbitrary values of m? In this case

(391) $\Delta\varphi = \varphi(m) - \varphi_0 = -k_BT\ln F_S(g\,\mu\beta\lambda m)$

$F_S(x) = \frac{\sinh\left[\frac{2S+1}{2}x\right]}{\sinh\left[\frac{x}{2}\right]}\cdot\frac{1}{(2S+1)}$; We have changes relative to the previous expression since $\mu = g\mu_B S$ was containing an S which has now been explicitly taken into account

$x = g\mu_B\lambda m$

Using a computer it is easy to verify that $F_S(x) > 1$ for arbitrary $S = 1/2, 1, 3/2, \ldots$. Hence the ferromagnetic state is stable for the whole $m>0$ region.

IV. The magnetic susceptibility.

Now one takes the order parameter equation (371) with external magnetic field

(392) $m = \mu B_S(\mu\beta[H+\lambda m])$

where we place ourself close to T_c, so $m \ll 1$, and also small H. Using the B_S function expansion for small argument (372) up to first order in the argument one finds (from (392)):

(393) $m = \frac{S+1}{3S}\mu^2\beta\lambda m + \frac{S+1}{3S}\mu^2\beta H + O\left[(H+\lambda m)^3\right]$

But since from (378), $k_BT_c = \frac{S+1}{3S}\lambda\mu^2$, from (393) one has

(394) $m\left(1-\frac{T_c}{T}\right) = \frac{S+1}{3S}\mu^2\beta H$; Now the derivative in function of H by taking into account that $\chi_1 = \frac{\partial m}{\partial H}$ leads to

(395) $\chi_1\left(\frac{T-T_c}{T}\right) = \frac{S+1}{3S}\mu^2\beta$; $\chi_1 = \frac{\chi}{N}$, because $M = Nm$; Now

(396) $\chi = N\frac{S+1}{3S}\mu^2\beta\frac{T}{T-T_c} = N\frac{S+1}{3S}\frac{1}{k_B}\mu^2\frac{1}{T-T_c}$; $\mu = g\mu_B S$

Denoting by

(397) $C = N(g\mu_B)^2\frac{S(S+1)}{3k_B}$; = Curie constant, one obtains

(398) $\chi = \frac{C}{T-T_c}$; which is the Curie-Weiss law; Furthermore T_c = Curie temperature.

Since $\chi \sim t^{-1}$; and $\chi \sim t^{-\gamma}$ is the definition of the γ crit. exponent, one finds

(399) $\boxed{\gamma = 1}$

Furthermore from (398) one knows that $1/\chi$ plotted in function of T gives a line, see Fig 29,

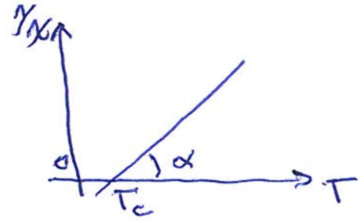

Fig 29

and experimental data like in Fig 29 indicate that at T_c ferromagnetis occurs. Furthermore, if $\lambda = 0$ and $T_c = 0$, the χ has the form

(400) $\chi = \dfrac{C}{T}$; $tg\,\alpha = C$

which is called the Curie law which provide χ for paramagnets. In such a case the line in Fig 29 crosses the origine. Note that from the slope of the line in Fig 29 the Curie constant can be obtained from the experimental data. It allows to find experimentally the effective magnetic moment (see (397)):

(401) $\mu_{eff} = g\mu_B\sqrt{S(S+1)}$; since $C = \dfrac{N}{3k_B}\mu_{eff}^2$;

Furthermore, from $T_c = \dfrac{S(S+1)}{3k_B} J\cdot z$, the interspin coupling J can be deduced.

V. Field dependence of the order parameter.

In order to find the field dependence of the order parameter, we must use also the third order term from the Taylor expansion of the $B_S(x)$ function around $x=0$ from (372) in equation for the order parameter (392). In this way instead of (393) we find

(402) $$m = \frac{S+1}{3S}\mu^2\lambda\beta m + \frac{S+1}{3S}\mu^2 H\beta - \frac{S+1}{3S}\,\frac{2S^2+2S+1}{30S^2}\,\mu\left(\mu\beta[H+\lambda m]\right)^3 + O\left((H+\lambda m)^5\right)$$

Then, because of the T_c expression, ($k_B T_c = \frac{S+1}{3S}\lambda\mu^2$), the first term in the right side of (402) can be moved to the left as in (394), and since $H\to$ is considered, H is neglected from the last term in the right side. One finds:

(403) $$m\left(1-\frac{T_c}{T}\right) = \frac{S+1}{3S}\mu^2 H\beta - Km^3 \ ; \quad K = \frac{S+1}{3S}\cdot\frac{2S^2+2S+1}{30S^2}\mu^4\lambda^3\beta_c^3$$

We would like to arrive to the $\Delta \sim h^{1/\delta}$; $t=0$ relation, hence in the constant K, $\beta = \beta_c = \frac{1}{k_B T_c}$ has been taken ($m \sim H^{1/\delta}$ in our case). Taking now $T = T_c$ (i.e. $t=0$), (403) becomes of the form

(404) $$\frac{S+1}{3S}\mu^2\beta_c\cdot H = K\cdot m^3 \implies m = \left[\frac{S+1}{3S}\mu^2\frac{\beta_c}{K}\right]^{1/3}\cdot H^{1/3}$$

from where is seen that for the critical exponent δ one has

(405) $$\boxed{\delta = 3}$$

VI. The specific heat critical exponent.

Based on its definition, $C \sim |t|^{-\alpha}$, $h=0$, so we analyze the system without external magnetic field. In this case, based on (368) ($\hat{H} = -g\mu_B(\vec{H}+\lambda\vec{m})\sum_i \vec{S}_i$) becomes of the form

(406) $$\hat{H} = -g\mu_B\lambda\vec{m}\cdot\sum_i\vec{S}_i$$

But $\langle\hat{H}\rangle = U = -g\mu_B\lambda\vec{m}\langle\sum_i\vec{S}_i\rangle$ is the energy of the system. Since the magnetisation (see (364)) is $\vec{M} = g\mu_B\sum_i\langle\vec{S}_i\rangle = \vec{m}\cdot N \implies$

(407) $$\sum_i\langle\vec{S}_i\rangle = \frac{N}{g\mu_B}\vec{m}.$$

Introducing (407) in (406), we find

(408) $U = -\lambda N m^2$

For $t \to 0$, one has from (383) $m^2 \sim |t|$, hence $U \sim |t|$. Because $C = \frac{\partial U}{\partial T}$ and T enters linearly in U, C will not have T dependence since becomes a constant. Divergence is not present, so one has

(409). $\boxed{\alpha = 0}$

Now one has all critical exponents characterising the mean-field theory from (384, 399, 405, 409). The critical exponents related to the correlation function are borrowed from (333, 335) of Lecture 6.

The summary of the mean-field results is:

(410) $\alpha = 0, \beta = \frac{1}{2}, \gamma = 1; \delta = 3; \nu = \frac{1}{2}; \eta = 0;$

VII. The order parameter behavior at $T \to 0$.

However is not connected to critical exponents, it is useful in this stage to analyze the $T \to 0$ behavior of the magnetisation, since in this manner one finds information on the behavior of the order parameter on the entire domain $T < T_c$.

In order to perform this calculation we need the large argument expansion of $B_S(x)$ because $x = \mu\beta\lambda m \to \infty$ when $T \to 0$. The large argument expansion for $\coth x$ is

(411) $\coth x = \frac{1}{\tanh x} = 1 + 2e^{-2x} + \dots ; \; x \gg 1$

From (411) the large argument expansion of the Brillouin function becomes

(412) $B_S(x) = 1 - \frac{1}{S} \underbrace{\exp\left[-\frac{x}{S}\right]}_{y} + O(y^{2S+1})$

From (371) where one has $m = \mu B_S(\mu\lambda\beta m)$, the $m(T=0) = m_0 = \mu = g\mu_B S$ is obtained for the m value at $T=0$, hence one finds $\left(k_B T_c = \frac{S+1}{3S}\lambda\mu^2\right)$

(413) $B_S(\mu\beta\lambda m)\Big|_{T\to 0} = 1 - \frac{1}{S}\exp\left(-\frac{\mu^2\lambda}{S}\beta\right) = 1 - \frac{1}{S}\exp\left[-\frac{3}{S+1}\frac{T_c}{T}\right]$

Consequently, in the $T \to 0$ limit one has from (371)

(414) $m = \mu - \frac{\mu}{S}\exp\left[-\frac{3}{S+1}\frac{T_c}{T}\right] + O\left[\left(\exp\left(-\frac{3}{S+1}\frac{T_c}{T}\right)\right)^{2S+1}\right]$

Calculating the derivative of (414) in function of T one finds

(415) $\frac{\partial m}{\partial T} = -\frac{3T_c\mu}{S(S+1)}\frac{1}{T^2}\exp\left[-\frac{3}{S+1}\frac{T_c}{T}\right]$

which tends to zero at $T \to 0$, i.e.

(416) $\lim_{T\to 0}\frac{\partial m}{\partial T} = 0$; Taking now into account (383, 416) and connecting numerically the $T \to 0$ and $T \to T_c$ regions one finds the qualitative behavior of the order parameter $\Delta = m$, see Fig 30

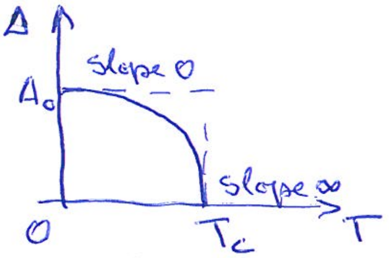

Fig 30.

Note that this qualitative behavior remains true also outside the mean-field. Evidently, in this case β changes, but the the rough behaviour remains the same (at least for continuous transitions).

VIII. Some words about mean-field.

Comparing the critical exponent of the mean-field theory (410) to experimental data (e.g. the Gugenheim measurements provide $\beta = \frac{1}{3}$ for real gases), one sees that in fact mean-field is a poor approximation. In practice, mean-field overestimates the ordering tendencies providing often with 50% higher T_c. Nowadays, when we analyze strongly correlated systems since these are of interest today, and we know that in mean-field the correlations are neglected, we often accentuate the poorness of the mean-field.

But I would like to mention that a century ago mean-field was a real challange, and the work of several Nobel prize winners was collected here in order to present you this material (Pieter Zeeman, Pierre Curie, van der Waals, David Bohr, Werner Heisenberg).

In the context of the scaling laws, the mean-field exponents (410) will be analyzed further on.

IX. Some words about dynamical mean-field theory:

Dynamical mean-field is a complicated numerical procedure which is done in $d=\infty$. The calculations provided in Fourier transformed variables contain only ω dependences, while k-dependences are missing. This means that only correlations in time are taken into account, but correlations in $\vec{r}$ space are neglected. It is a step forward from the standard mean-field which neglects all correlations and it is also valid (see the upper critical dimension notion) in high dimensions. During the technic the many-body lattice problem is mapped into a many-body local (so called "impurity") problem, that is why the correlations in space are missing. The improvement in critical exponents values has not been analyzed to date.